again the idea: transformations that leave $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

form invariant: $(q,p) \to (\underline{Q},\underline{P})$

$$H(q,p,t) \to \bar{H}(Q,\underline{P},t)$$

variable transform: $\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_f \\ p_1 \\ \vdots \\ p_f \end{pmatrix}$, $\underline{y} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_f \\ P_1 \\ \vdots \\ P_f \end{pmatrix}$ $\qquad \dot{\underline{x}} = \underline{\underline{J}} H_x$

$$\Rightarrow M_{\alpha\beta} = \frac{\partial x_\alpha}{\partial y_\beta}, \quad M^{-1}_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}$$

$$\Rightarrow \underline{\underline{M}} = \underline{\underline{J}}\left(\underline{\underline{J}}\,\underline{\underline{M}}^{-1}\right)^T \Leftrightarrow \underline{\underline{M}}^T \underline{\underline{J}}\,\underline{\underline{M}} = \underline{\underline{J}}, \quad \underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \mathbb{1} \\ -\mathbb{1} & \underline{\underline{0}} \end{pmatrix}$$

- $\underline{\underline{J}}$ defines a skew-symmetric (non-degenerate, bilinear) form / alternating form : $(\underline{u},\underline{v}) = -(\underline{v},\underline{u}) = \underline{u}^T \underline{\underline{J}}\,\underline{v}$

- $(\underline{u},\underline{u}) = 0 \Rightarrow 0 = \{\underline{u}+\underline{w}, \underline{u}+\underline{w}\} = \underbrace{\{\underline{u},\underline{u}\}}_{=0} + \{\underline{w},\underline{u}\} + \{\underline{u},\underline{w}\} + \underbrace{\{\underline{w},\underline{w}\}}_{=0}$

$$\Rightarrow \{\underline{w},\underline{u}\} = -\{\underline{u},\underline{w}\}$$

- $\left.\begin{aligned} \dot{y}_j &= \sum_{k=1}^{2f} \frac{\partial y_j}{\partial x_k}\dot{x}_k \\ \frac{\partial H}{\partial y_j} &= \sum_{k=1}^{2f} \frac{\partial H}{\partial x_k}\frac{\partial x_k}{\partial y_j} \end{aligned}\right\} \quad \begin{aligned} &\dot{\underline{y}} = \underline{\underline{M}}^{-1}\dot{\underline{x}} = \dots = \underline{\underline{J}}\,\underline{H}_y \quad (\dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x) \\ &\underline{H}_y = \underline{\underline{M}}^T \underline{H}_x \end{aligned}$

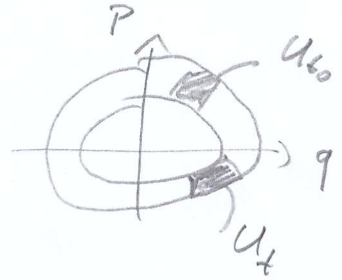
- $\underline{\underline{M}} = \underbrace{\begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{pmatrix}}_{2f} \Big\} 2f \longleftrightarrow \begin{pmatrix} M_3(p,Q,t) & M_4(p,\underline{P},t) \\ M_1(q,Q,t) & M_2(q,\underline{P},t) \end{pmatrix}$

Def: The set of matrices $\underline{\underline{M}}$ (canonical transforms) with $\underline{\underline{M}}^T \underline{\underline{J}}\,\underline{\underline{M}} = \underline{\underline{J}}$ forms the (real) symplectic group S over $\mathbb{R}^{2f}$.

§5.4 Der Liouvillesche Satz

Satz: Bei der Hamiltonschen Zeitentwicklung ist das Phasenraumvolumen erhalten.

Phasenraumvolumen:



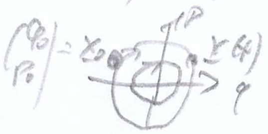
Hamiltonsche Zeitentwicklung: $\dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x$

$$\hookrightarrow \quad \underline{x}(t, \underbrace{t_0, \underline{x}_0}_{\text{Anfangsbed.}}) =: \underline{\phi}_{t,t_0}(\underline{x}_0)$$ Fluss im Phasenraum

$$\left.\begin{array}{l} \underline{\phi}_{t,t_0}: \Gamma \longrightarrow \Gamma \\ \underline{x}_0(t_0) \longmapsto \underline{x}(t) \end{array}\right\}$$ beschreibt die Zeitentwicklung der Anfangskonfigurationen

Bsp.: 1D harmonischer Oszillator:

$$(q,p) \in \Gamma, \quad \left.\begin{array}{l} \dot{q} = p \\ \dot{p} = -\omega_0^2 q \end{array}\right\} \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$



$$\Rightarrow \underline{x}(t) = \phi_{t,t_0}(\underline{x}_0) = \exp\left((t-t_0)\underline{\underline{A}}\right)\underline{x}_0$$

$$= \left(\sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!}\underline{\underline{A}}^n\right)\underline{x}_0$$

$$\underline{\underline{A}}^2 = -\omega_0^2 \underline{\underline{\mathbb{1}}}$$
$$\underline{\underline{A}}^{2n} = (-1)^n \omega_0^{2n} \underline{\underline{\mathbb{1}}}$$
$$\underline{\underline{A}}^{2n+1} = (-1)^n \omega_0^{2n+1} \frac{1}{\omega_0}\underline{\underline{A}}$$

$$= \left(\underline{\underline{\mathbb{1}}}\cos(\omega_0(t-t_0)) + \frac{\underline{\underline{A}}}{\omega_0}\sin(\omega_0(t-t_0))\right)\underline{x}_0$$

$$= \underbrace{\begin{pmatrix} \cos(\omega_0(t-t_0)) & \frac{1}{\omega_0}\sin(\omega_0(t-t_0)) \\ -\omega_0\sin(\omega_0(t-t_0)) & \cos(\omega_0(t-t_0)) \end{pmatrix}}_{\underline{\underline{P}}(t-t_0)} \underline{x}_0$$

Beweis: Menge von Anfangskonfigurationen (x_0) mit Phasenraumgebiet U_{t_0} mit Volumen $V_{t_0} = \int_{U_{t_0}} d^{2f} x_0$

$$\Rightarrow \quad V_t = \int_{U_t} d^{2f} x = \int_{U_{t_0}} d^{2f} x_0 \, \det\left(\frac{\partial x}{\partial x_0}\right) = \int_{U_{t_0}} d^{2f} x_0 \, \det\left(D\phi_{t,t_0}(x_0)\right)$$

Funktionaldeterminante

$$\left(D\phi_{t,t_0}(x_0)\right)_{ik} := \frac{\partial \phi^i_{t,t_0}(x_0)}{\partial x_0^k} = \frac{\partial x^i}{\partial x_0^k}, \quad \text{Jacobi-Matrix}$$

- $t \approx t_0$: $\phi_{t,t_0}(x_0) = x_0 + \underbrace{F(x_0, t)}_{\underline{\underline{J}}\,\underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}} (t - t_0) + O((t-t_0)^2)$

$$\Rightarrow \quad \frac{\partial \phi^i_{t,t_0}(x_0)}{\partial x_0^k} = \delta_{ik} + \frac{\partial F^i}{\partial x_0^k}(t-t_0) + O((t-t_0)^2)$$

mit $\det(\mathbb{1} + \underline{\underline{B}}\varepsilon) = 1 + \varepsilon \, \mathrm{Sp}(\underline{\underline{B}}) + O(\varepsilon^2)$:

$$\det(D\phi_{t,t_0}) = 1 + (t-t_0) \underbrace{\sum_{i=1}^{2f} \frac{\partial F^i}{\partial x_0^i}}_{\mathrm{div}\,\underline{F} = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial}{\partial p}\frac{\partial H}{\partial q} = 0}$$

$$\rightarrow \quad V_t = \int_{U_{t_0}} d^{2f} x_0 \left(1 + O(t-t_0)^2\right) \approx V_{t_0}$$

Bsp.: 1D harmonische Osz.

$$x^i(t) = \sum_{k=1}^{2} P_{ik} \, x_0^k \quad \text{mit } \det \underline{P} = \cos^2(\omega_0(t-t_0)) + \sin^2(\omega_0(t-t_0)) = 1$$

$$\Rightarrow \quad \underbrace{dx^1 dx^2}_{\hookrightarrow \int_{U_t} d^{2f}x = \int_{U_{t_0}} d^{2f}x_0} = (\det \underline{P}) \, dx_0^1 dx_0^2 = dx_0^1 dx_0^2$$

5.5 Poisson-Klammern

Jede **Observable** (phys. Größe, Energie, Impuls ...) lässt sich als Funktion von q, p, t schreiben: $g(q, p, t)$

$\Rightarrow$ zeitliche Änderung längs der Bahn $q(t), p(t)$ im Phasenraum Γ:

$$\frac{dg}{dt} = \sum_{i=1}^{f} \left(\frac{\partial g}{\partial q_i} \dot{q}_i + \frac{\partial g}{\partial p_i} \dot{p}_i \right) + \frac{\partial g}{\partial t}$$

$$= \sum_{i=1}^{f} \left(\frac{\partial g}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial g}{\partial t}$$

$$=: \{g, H\} + \frac{\partial g}{\partial t}$$

Def: Für zwei beliebige Observable $g(q, p, t)$, $h(q, p, t)$ heißt

$$\{g, h\} := \sum_{i=1}^{f} \left(\frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right)$$ **Poisson-Klammer**

- $\{g, h\} = (\underline{g}_x, \underline{h}_x) = \underline{g}_x^T \underline{J}\, \underline{h}_x = \sum_{i,k} \frac{\partial g}{\partial x_i} J_{ik} \frac{\partial h}{\partial x_k}$

$$= \begin{pmatrix} \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{pmatrix} \begin{pmatrix} \underline{0} & \mathbb{1} \\ -\mathbb{1} & \underline{0} \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial q} \\ \frac{\partial h}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial p} \\ -\frac{\partial h}{\partial q} \end{pmatrix}$$

- $\{g, h\} = -\{h, g\}$ schiefsymmetrisch

  $\{g, \lambda_1 h_1 + \lambda_2 h_2\} = \lambda_1 \{g, h_1\} + \lambda_2 \{g, h_2\}$

  $\{g, h\} = 0 \quad \forall h \Rightarrow g = \text{const.}$

  $\{g, g\} = 0$

  $\Big\}$ Symplektisches Skalarprodukt!

Satz: Die Poisson-Klammern sind invariant unter kanonischen Trafos:

Beweis: $\frac{\partial g}{\partial x_i} = \sum_k \frac{\partial g}{\partial y_k} \frac{\partial y_k}{\partial x_i} = \sum_k M^{-1}_{ki} \frac{\partial g}{\partial y_k} \Rightarrow \underline{g}_x = (\underline{\underline{M}}^{-1})^T \underline{g}_y$

$\Rightarrow \underline{g}_x^T = \underline{g}_y^T (\underline{\underline{M}}^{-1})$

$$\{g,h\} = (\underline{g}_x, \underline{h}_x) = \underline{g}_x^T \underline{\underline{J}}\, \underline{h}_x = \underline{g}_y^T \underbrace{\underline{\underline{M}}^{-1} \underline{\underline{J}} (\underline{\underline{M}}^{-1})^T}_{\underline{\underline{J}}} \underline{h}_y$$

$$= \underline{g}_y^T \underline{\underline{J}}\, \underline{h}_y = (\underline{g}_y, \underline{h}_y) = \{g,h\}$$

Satz: Die Trafo $(q,p) \longrightarrow (Q,P)$ ist genau dann kanonisch, wenn $\{Q_k, P_j\} = \delta_{kj}$

$\{Q_i, Q_j\} = \{P_i, P_j\} = 0$