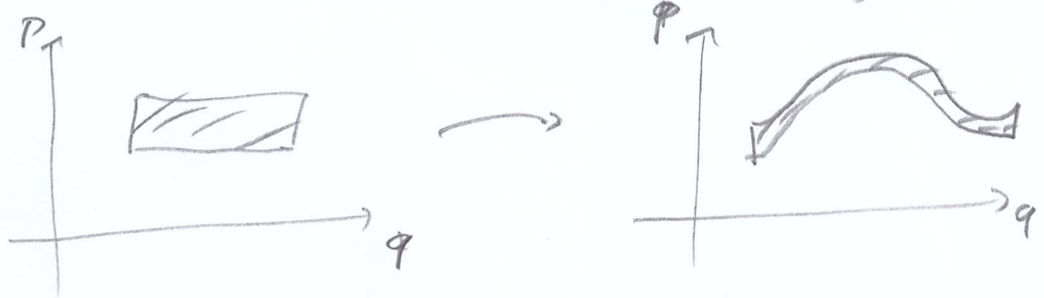
5.4 Liouville's theorem

Thm: The flow of Hamiltonian systems preserves the phase space volume.

(The phase-space distribution function is constant along the trajectories of Hamiltonian systems.)



Same area!

- flow: $\phi_{t,t_0} : \Gamma \to \Gamma$

$$\underline{x}_0(t_0) \mapsto \phi_{t,t_0}(x_0) = \underline{x}(t)$$

5.5 Poisson brackets

given 2 functions/observables $g(q,p,t)$ and $h(q,p,t)$, their Poisson brackets takes the form:

$$\{g,h\} = \sum_{i=1}^{f} \left(\frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right)$$

- $\{g,h\} = -\{h,g\}$
- $\{g, \lambda_1 h_1 + \lambda_2 h_2\} = \lambda_1 \{g, h_1\} + \lambda_2 \{g, h_2\}$
- $\{g,h\} = 0 \ \forall h \Rightarrow g = \text{const.}$
- $\{g,g\} = 0$
- $\frac{dg}{dt} = \{g, H\} + \frac{\partial g}{\partial t}$
- $\{g,h\} = (\underline{g}_x, \underline{h}_x) = \underline{g}_x^T \underline{\underline{J}}\, \underline{h}_x$, $\underline{g}_x = \begin{pmatrix} \frac{\partial g}{\partial q_1} \\ \vdots \\ \frac{\partial g}{\partial p_f} \end{pmatrix}$, $\underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \mathbb{1} \\ -\mathbb{1} & \underline{\underline{0}} \end{pmatrix}$, $\underline{h}_x = \begin{pmatrix} \frac{\partial h}{\partial q_1} \\ \vdots \\ \frac{\partial h}{\partial p_f} \end{pmatrix}$

5.5 Poisson-Klammern (Fortsetzung)

Eigenschaften:

$g = g(q,p)$: nicht explizit zeitabhängig $\left(\frac{\partial g}{\partial t} = 0\right)$

$$\Rightarrow \frac{dg}{dt} = \{g, H\}$$

- g : Bewegungskonstante: $\left(\frac{dg}{dt} = 0\right) \Leftrightarrow \{g, H\} = 0$

- $g = q_k$ oder $g = p_k$

$$\dot{q}_k = \{q_k, H\} = \sum_{i=1}^{f} \Big(\underbrace{\frac{\partial q_k}{\partial q_i}}_{\delta_{ik}} \frac{\partial H}{\partial p_i} - \underbrace{\frac{\partial q_k}{\partial p_i}}_{=0} \frac{\partial H}{\partial q_i} \Big) = \frac{\partial H}{\partial p_k}$$

$$\dot{p}_k = \{p_k, H\} = \sum_{i=1}^{f} \Big(\underbrace{\frac{\partial p_k}{\partial q_i}}_{=0} \frac{\partial H}{\partial p_i} - \underbrace{\frac{\partial p_k}{\partial p_i}}_{\delta_{ik}} \frac{\partial H}{\partial q_i} \Big) = -\frac{\partial H}{\partial q_k}$$

Kompakt: $\dot{x}_k = \{x_k, H\} = \sum_{i,j=1}^{2f} \underbrace{\frac{\partial x_k}{\partial x_i}}_{\delta_{ik}} J_{ij} \frac{\partial H}{\partial x_j} = \sum_{j=1}^{2f} J_{kj} \frac{\partial H}{\partial x_j}$

oder $\underline{\dot{x}} = \underline{\underline{J}}\, \underline{H}_x$

- Fundamentale Poisson-Klammern

$$\left.\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned}\right\} \quad \{x_i, x_j\} = \sum_{k,l=1}^{2f} \underbrace{\frac{\partial x_i}{\partial x_k}}_{\delta_{ik}} J_{kl} \underbrace{\frac{\partial x_j}{\partial x_l}}_{\delta_{jl}} = J_{ij}$$

- Satz: Die Transformation $(q,p) \longrightarrow (Q,P)$ ist genau dann kanonisch, wenn $\{Q_i, P_j\} = \delta_{ij}$, $\{Q_i, Q_j\} = 0 = \{P_i, P_j\}$.

Beweis: $\left.\begin{aligned} &\bullet\ \underline{\dot{x}} = \underline{\underline{J}}\, \underline{H}_x \\ &\bullet\ \underline{\underline{M}}^T \underline{\underline{J}}\, \underline{\underline{M}} = \underline{\underline{J}} \\ &\bullet\ \underline{\underline{M}} = \underline{\underline{J}} (\underline{\underline{J}}\, \underline{\underline{M}})^T \end{aligned}\right\} \Longleftrightarrow \dot{y}_k = \sum_{l=1}^{f} J_{kl} \frac{\partial H}{\partial y_l} \Leftrightarrow \{y_k, y_l\} = J_{kl}$

$M_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}$, $\underline{\dot{y}} = \underline{\underline{M}}\, \underline{\dot{x}}$

Beweis als HA :)

Fazit:

Folgende Aussagen sind äquivalent:

1. Die Transformation $\mathbf{x} \mapsto \mathbf{y}$ ist kanonisch.
2. Die Gleichungen $\dot{\mathbf{x}} = \mathbf{J}\mathbf{H}_x$ sind invariant.
3. Die Poisson-Klammern $\{g, h\}$ sind invariant für alle g, h.
4. Die fundamentalen Poisson-Klammern $\{x_i, x_j\} = \delta_{ij}$ sind invariant.
5. Die Jacobi-Matrix $M_{\alpha\beta} = \frac{\partial x_\alpha}{\partial y_\beta}$ ist symplektisch, d.h.: $\mathbf{M}^T\mathbf{J}\mathbf{M} = \mathbf{J}$.
6. Es existiert eine Erzeugende.

Bezug zur Quantenmechanik:

Hilbert-Raum

Klass. Observable: $g(q,p,t)$ $\longrightarrow$ qm Operator $\hat{g}: \mathcal{H} \to \mathcal{H}$

Poisson-Klammer $\{f,g\}$ $\longrightarrow$ $\frac{1}{i\hbar}[\hat{f},\hat{g}] := \frac{1}{i\hbar}(\hat{f}\hat{g} - \hat{g}\hat{f})$

Kommutator

Fundamentale Poissonklammern

$\{q_i, p_j\} = \delta_{ij}$ $\longrightarrow$ $[\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$

$\{q_i, q_j\} = \{p_i, p_j\} = 0$ $\longrightarrow$ $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

Hamilton-Funktion $H(q,p,t)$ $\rightarrow$ Hamilton-Operator: $\hat{H}(\hat{q},\hat{p})$

Bewegungsgleichungen: Heisenberg'sche Bewegungsgleichungen

$$\frac{dg}{dt} = \{g, H\} + \frac{\partial g}{\partial t} \quad \rightarrow \quad \frac{d}{dt}\hat{g} = \frac{1}{i\hbar}[\hat{g}, \hat{H}] + \frac{\partial \hat{g}}{\partial t}$$

5.6 Hamilton-Jacobi-Theorie

Idee: Finde kanonische Trafo, die alle Koordinaten zyklisch macht.

etwa (einfachster Fall): $\bar{H} \equiv 0$

wähle: $M_2(q,P,t) =: S(q,P,t)$ $\quad (q,p) \to (Q,P)$

$$H(q,p,t) \to \bar{H}(Q,P,t) = H + \frac{\partial S}{\partial t}$$

$$\text{mit } p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial P_k}$$

so dass: $\bar{H} = H\left(q_1, \dots, q_f, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_f}, t\right) + \frac{\partial S}{\partial t} = 0$

Hamilton-Jacobi-Gleichung

$$\Rightarrow \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k} = 0 \quad \Rightarrow Q_k = \text{const} = \beta_k$$

Parameter!

$$\dot{P}_k = -\frac{\partial \bar{H}}{\partial Q_k} = 0 \quad \Rightarrow P_k = \text{const} = \alpha_k \Rightarrow S(q,\alpha,t)$$

Kochrezept:

(i) $H(q,p,t)$, $p_k = \frac{\partial S}{\partial q_k} \Rightarrow H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$

(ii) Löse Hamilton-Jacobi-Gleichung $\Rightarrow S(q,\alpha,t)$, $\alpha = P$

(iii) Aus Erzeugenden $S(q,\alpha,t)$ folgt: $Q_k = \frac{\partial S(q,\alpha,t)}{\partial \alpha_k} = \beta_k$

$$\Rightarrow q_j = q_j(\alpha,\beta,t)$$

(iv) $p_j = \frac{\partial S}{\partial q_j} = p_j(q,\alpha,t) = p_j(q(\alpha,\beta),\alpha,t)$

(v) Bestimmung von α, β aus Anfangsbedingungen

$$\left.\begin{aligned} q_j(0) &= q_j(\alpha,\beta,0) \\ p_j(0) &= p_j(\alpha,\beta,0) \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= \alpha(q(0),p(0)) \\ \beta &= \beta(q(0),p(0)) \end{aligned}$$

Bedeutung von S:

$$\frac{dS(q,\alpha,t)}{dt} \underset{\alpha=\text{const}}{\overset{\uparrow}{=}} \sum_{k=1}^{f} \frac{\partial S}{\partial q_k}\dot{q}_k + \underbrace{\frac{\partial S}{\partial t}}_{-H' = -H = H} = \sum_{k=1}^{f} p_k \dot{q}_k - H = L$$

$\Rightarrow$ $S = \int L\,dt$ Wirkungsfunktional entlang Trajektorie (noch unbekannt)

Bsp: 1D harmonischer Oszillator:

(i) $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$

$S(q,P,t)$ mit $p = \frac{\partial S}{\partial q}$

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{m}{2}\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

(ii) Ansatz: $S(q,P,t) = W(q;\underset{\uparrow}{P}) + V(t;P)$ Separationsansatz nach q,t.
Parameter

$$\Rightarrow \underbrace{\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{m}{2}\omega^2 q^2}_{\text{unabhängig von } t} = -\frac{dV}{dt} = \alpha = \text{const}$$

$\Rightarrow$ (a) $V(t) = -\alpha t + V_0$

(b) $\left(\frac{dW}{dq}\right)^2 = m^2\omega^2\left(\frac{2\alpha}{m\omega^2} - q^2\right) \Rightarrow \overset{W(q,\alpha)}{W} = m\omega\int dq\sqrt{\frac{2\alpha}{m\omega^2} - q^2}$

$$\Rightarrow S(q,\alpha,t) \overset{S=W+V}{=} m\omega\int dq\sqrt{\frac{2\alpha}{m\omega^2} - q^2} - \alpha t + \underset{\overset{!}{0}\text{ o.B.d.A.}}{V_0} = -\alpha t + m\omega\left[\frac{q}{2}\sqrt{\frac{2\alpha}{m\omega^2} - q^2} + \frac{\alpha}{m\omega^2}\arcsin\left(q\sqrt{\frac{m\omega^2}{2|\alpha|}}\right)\right]$$

(iii) $Q = \frac{\partial S}{\partial \underset{\alpha \equiv P}{\alpha}} = -t + \frac{1}{\omega}\int dq \frac{1}{\sqrt{\frac{2\alpha}{m\omega^2} - q^2}}$

$$= -t + \frac{1}{\omega}\arcsin\left(q\omega\sqrt{\frac{m}{2\alpha}}\right) \overset{!}{=} \beta = \text{const}$$

$[Q] = \text{Zeit}$, $\beta = t_0$

$$\Rightarrow \quad q = \frac{1}{\omega}\sqrt{\frac{2\alpha}{m}}\, \sin(\omega(t+\beta))$$

(iv) $$p = \frac{\partial S}{\partial q} = \frac{dW}{dq} = m\omega\sqrt{\frac{2\alpha}{m\omega^2} - q^2}$$

$$= \sqrt{2\alpha m}\, \cos(\omega(t+\beta))$$

(v) $t = t_0 = 0$, $p(0) = 0$, $q(0) = q_0 \neq 0$

$$\Rightarrow q_0 = \sqrt{\frac{2\alpha}{m\omega^2}}\, \sin(\omega\beta)$$

$$0 = p_0 = \sqrt{2\alpha m}\, \cos(\omega\beta) \Rightarrow \beta = \frac{\pi}{2\omega}$$

$$\Rightarrow q_0 = \sqrt{\frac{2\alpha}{m\omega^2}}$$

$\Rightarrow \alpha = \frac{m}{2}\omega^2 q_0^2 = E$ (max ausgelenkt $\Rightarrow$ max pot. Energie)

$\Rightarrow \alpha = P = E$ (Energie)

$Q = t_0$ (Zeit)

neue verallgem. Koordinaten

Speziell: $\frac{\partial H}{\partial t} = 0 \Leftrightarrow \frac{dH}{dt} = \{H, H\} = 0$, H: Integral der Bewegung.

$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0 \Rightarrow$ Ansatz: $S(q, P, t) = W(q; P) - Et$

$$\Rightarrow H\left(q, \frac{\partial W}{\partial q}\right) = E$$

- $P_j = \frac{\partial W}{\partial q_j}$, $Q_j = \frac{\partial W}{\partial P_j}$, $\bar{H} = H = E \Rightarrow \dot{Q}_j = \frac{\partial \bar{H}}{\partial P_j} = \frac{\partial E}{\partial \alpha_j} = \omega_j$

$$Q_j = \omega_j t + \beta_j = \frac{\partial W}{\partial \alpha_j}$$

W als Erzeugende
interpretiert als kanonische Trafo