5.5 Poisson brackets (continued)

- $g = g(q, p, t) \Rightarrow \frac{dg}{dt} = \{g, H\}$
- fundamental Poisson brackets:

$$\left.\begin{aligned} &\{q_i, q_j\} = 0 = \{p_i, p_j\} \\ &\{q_i, p_j\} = \delta_{ij} \end{aligned}\right\} \Leftrightarrow \{x_i, x_j\} = J_{ij} \qquad \underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{1}} \\ -\underline{\underline{1}} & \underline{\underline{0}} \end{pmatrix}$$

- Theorem: The transformation $(q, p) \to (\underline{Q}, \underline{P})$ is canonical

$$\text{iff } \{Q_i, P_j\} = \delta_{ij}, \ \{Q_i, Q_j\} = 0 = \{P_i, P_j\}.$$

$\to$ Proof?

- wrapup: equivalent statements:
  - (i) $\underline{x} \mapsto \underline{y}$ is canonical
  - (ii) $\dot{\underline{x}} = \underline{\underline{J}}\, \underline{H}_x$ are invariant.
  - (iii) $\{g, h\}$ are invariant
  - (iv) $\{x_i, x_j\} = J_{ij}$ are invariant
  - (v) Jacobian $M_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}$ is symplectic $(\underline{\underline{M}}^T \underline{\underline{J}}\, \underline{\underline{M}} = \underline{\underline{J}})$
  - (vi) There exists a generating function
- link to QM

5.6 Hamilton-Jacobi theory

idea: find canonical transform for which all variables become canonical.

$\hookrightarrow \bar{H} \equiv 0$

$\hookrightarrow M_2(q, \underline{P}, t) =: S(q, \underline{P}, t)$ with: $(q, p) \longrightarrow (\underline{Q}, \underline{P})$

$$H(q,p,t) \longrightarrow \bar{H}(\underline{Q}, \underline{P}, t) = H + \frac{\partial S}{\partial t}$$

$$p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial P_k}$$

$$\Rightarrow \quad \bar{H} = H\left(q_1, \dots, q_f, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_f}, t\right) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k} = 0 \qquad \Rightarrow Q_k = \beta_k = \text{const.}$$

"parameter"

$$\dot{P}_k = -\frac{\partial \bar{H}}{\partial Q_k} = 0 \qquad \Rightarrow P_k = \alpha_k = \text{const.} \Rightarrow S(q, \underline{\alpha}, t)$$

- recipe:

(i) $H(q,p,t)$, $\quad p_k = \frac{\partial S}{\partial q_k} \Rightarrow H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$

(ii) Solve Hamilton-Jacobi equation $\Rightarrow S(q, \underline{\alpha}, t)$; $\underline{\alpha} = \underline{P}$

(iii) $Q_k = \frac{\partial S}{\partial \alpha_k} = \beta_k \Rightarrow \quad q_j = q_j(\underline{\alpha}, \beta, t)$

(iv) $p_j = \frac{\partial S}{\partial q_j} = p_j(q, \underline{\alpha}, t)$

(v) Determine $\underline{\alpha}, \beta$ from initial conditions:

$$\left.\begin{aligned} q_j(0) &= q_j(\alpha, \beta, 0) \\ p_j(0) &= p_j(\underline{\alpha}, \beta, 0) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \underline{\alpha} &= \underline{\alpha}(q(0), p(0)) \\ \beta &= \beta(q(0), p(0)) \end{aligned}$$

- $S = \int L \, dt$ "action"
- $\frac{\partial H}{\partial t} = 0 \Leftrightarrow \frac{dH}{dt} = \{H, H\} = 0 \Rightarrow H$: Integral of motion $\Rightarrow S(q, \underline{P}, t) = W(q) - Et$

$H(q, \cancel{p}) = E$

$S(q,t) = W(q) - Et$ (Wirkungswellen)

↳ Phasengeschwindigkeit: $\underline{v} \sim \nabla W$

↳ Teilchenimpuls: $p = \nabla W$

$\Rightarrow H(q, \nabla W) = \frac{1}{2m} (\nabla W)^2 + V(q) = E$

Schrödinger-Gleichung: $\underbrace{\left(-\frac{\hbar^2}{2m}\Delta + V(x)\right)}_{\hat{H}} \psi(x) = E\,\psi(x)$

$q \to x$

$p \to \frac{\hbar}{i}\nabla$

$\psi(x) = e^{\frac{i}{\hbar} W(x)}$ Wellenfunktion

6. Mechanik des starren Körpers

Idee: ausgedehnte, starre Körper (vgl. deformierbare Medien)
Elastomechanik / Hydrodyn.

↳ Def: Starrer Körper:

(a) System von N Massepunkten mit festen Abständen
(↳ Zwangsbedingungen) $M=\sum_{i=1}^{N} m_i$

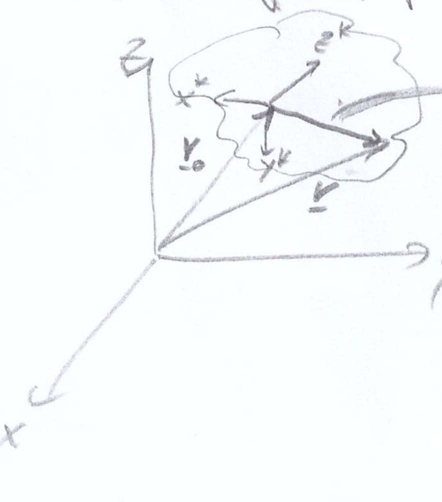
(b) vorgegebene, kontinuierliche Massenverteilung: $\rho(\underline{r})$, (Übergang: Dichte $\mu(r)$)
(↳ Gesamtmasse: $M=\int \rho(\underline{r})\, d^3r$)

Beschreibung: (i) Raumfestes Koordinatensystem $S:(x,y,z),(\underline{e}_x,\underline{e}_y,\underline{e}_z)$, $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$
↳ Inertialsystem

(ii) Körperfestes Koordinatensystem $S^*:(x^*,y^*,z^*),(\underline{e}_x^*,\underline{e}_y^*,\underline{e}_z^*)$, $(\underline{e}_1^*,\underline{e}_2^*,\underline{e}_3^*)$
↳ i. Allgemeinen / kein Inertialsystem
↳ Ursprung (häufig): Schwerpunkt (kann gedacht sein)
(auch beliebiger Bezugspunkt)

((iii) Schwerpunktsystem $S':(x',y',z'),(\underline{e}_x',\underline{e}_y',\underline{e}_z')$, $(\underline{e}_1',\underline{e}_2',\underline{e}_3')$
↳ translatorisch von S (gleiche Richtung))

\# Freiheitsgrade: $f=6$ (vgl. 2.1)



$\underline{r}-\underline{r}_0=\underline{d}$: Verschiebung (relativ zum Aufpunkt)
$\underline{d}\equiv\underline{r}^* \Rightarrow |\underline{d}| = \text{const}$ (Starrer Körper: feste Abstände!)
häufig: $\underline{r}_0$: Schwerpunktskoordinate $\underline{r}_s$

$\Rightarrow \underline{r} = \underline{r}_0 + \underline{d}$

$|\underline{d}| = \text{const}$ nur Drehung!

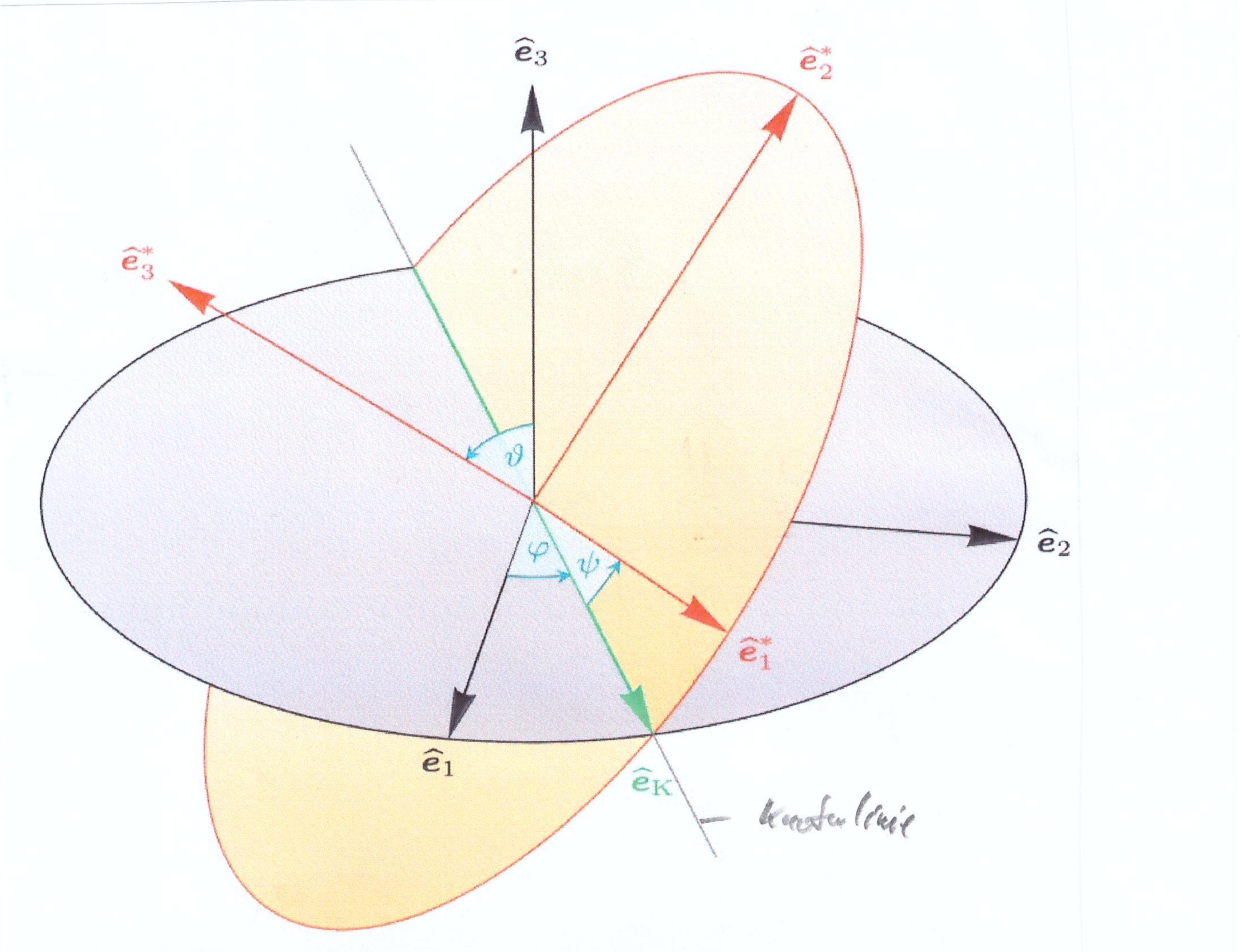
$\Rightarrow \dot{\underline{r}} = \dot{\underline{r}}_0 + \dot{\underline{d}} = \dot{\underline{r}}_0 + \underline{\omega}\times\underline{d}$

$\dot{\underline{r}}_0$: Schwerpunktsbew. / Aufpunkts

$\underline{\omega}\times\underline{d}$: Rotation um Drehachse gegeben durch $\underline{\omega}$

Bezeichg: $S \to S^*$: Translation $S \to S'$, Rotation $S' \to S^*$

Euler'sche Winkel: zur Beschreibung einer beliebigen Drehung $S' \to S^*$



φ: Winkel zwischen $\underline{e}_1$ und $\underline{e}_K$

ϑ: Winkel zwischen $\underline{e}_3$ und $\underline{e}_3^*$

ψ: —— " —— $\underline{e}_1^*$ und $\underline{e}_K$

⇒ (i) Drehung um φ (z-Achse / $\underline{e}_3$)

$$\underset{\substack{\uparrow\\ \text{z-Achse}}}{\underline{R}_3}(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

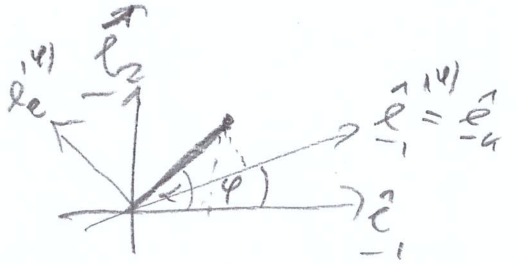
⇒ Basissystem: $\underbrace{\underline{e}_1^{(\varphi)}}_{=\underline{e}_K}, \underline{e}_2^{(\varphi)}, \underline{e}_3^{(\varphi)} = \underline{e}_3$

(ii) Drehung um ϑ (Knotenlinie, $\underline{e}_1^{(\varphi)} = \underline{e}_K$)

$$\underset{\substack{\uparrow\\ \text{x-Achse im }\varphi\text{-System}}}{\underline{R}_1}(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

⇒ Basissystem: $\underline{e}_1^{(\vartheta)}, \underline{e}_2^{(\vartheta)}, \underline{e}_3^{(\vartheta)}$

$\underline{e}_1^{(\vartheta)} = \underline{e}_1^{(\varphi)} = \underline{e}_K$, $\quad \underline{e}_3^{(\vartheta)} = \underline{e}_3^*$



Drehung um φ (z-Achse)

$$\left.\begin{aligned} x_1 &= r\cos\alpha \\ x_2 &= r\sin\alpha \end{aligned}\right\} \Rightarrow \begin{aligned} x_1' &= r\cos(\alpha-\varphi) = r(\cos\alpha\cos\varphi + \sin\alpha\sin\varphi) \\ x_2' &= r\sin(\alpha-\varphi) = r(\sin\alpha\cos\varphi - \cos\alpha\sin\varphi) \end{aligned}$$

$$\Rightarrow \begin{aligned} x_1' &= x_1\cos\varphi + x_2\sin\varphi \\ x_2' &= -x_1\sin\varphi + x_2\cos\varphi \end{aligned}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(iii) Drehung um ψ $\left(\underline{e}_3^* = \underline{e}_3^{(\vartheta)}\right)$

$$\underset{\substack{\uparrow \\ \text{z-Achse} \\ \text{im } \vartheta\text{-System}}}{\underline{\underline{R}}_3}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Kombiniert: $\underline{\underline{R}} = \underline{\underline{R}}_3(\psi)\, \underline{\underline{R}}_1(\vartheta)\, \underline{\underline{R}}(\varphi)$

$\Rightarrow$ Angewandt auf $\underline{d}$ in S nach $\underline{d}^*$ in S^*: $\underset{\substack{\uparrow \\ \text{Körperfest}}}{\underline{d}^*} = \underline{\underline{R}}\, \underset{\substack{\uparrow \\ \text{raumfest}}}{\underline{d}}$, $(\underline{r}^* = \underline{\underline{R}}\, \underline{r})$

Achtung: von $\underline{d}^*$ in $\underline{S}^*$ nach $\underline{d}$ in S:
$(\underline{d} = \underline{\underline{R}}^T \underline{d}^*,\ \underline{r} = \underline{\underline{R}}^T \underline{r}^*)$

$$\left.\begin{matrix} \varphi \to -\varphi \\ \vartheta \to -\vartheta \\ \psi \to -\psi \end{matrix}\right\} \quad \sin(\alpha) = -\sin(-\alpha)$$

$$\Rightarrow \underline{\underline{R}}_3(\varphi)^{-1} = \underline{\underline{R}}_3(\varphi)^T = \underline{\underline{R}}_3(-\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

analog: $\underline{\underline{R}}_1(\vartheta)^{-1} = \underline{\underline{R}}_1(\vartheta)^T = \underline{\underline{R}}_1(-\vartheta)$

6.1 Kinetische Energie & Trägheitstensor

(a) System von N Massepunkten: wähle: $r_0 = r_S$

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\underline{r}}_i^{2} = \frac{1}{2} \sum_{i=1}^{n} m_i \left[\frac{d}{dt}\left(\underline{r}_0 + \underline{d}^{(i)}\right)\right]^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \left[\dot{\underline{r}}_0 + \underline{\omega} \times \underline{d}^{(i)}\right]^2$$

$$= \frac{1}{2} \underbrace{\sum_{i=1}^{N} m_i}_{M} \dot{r}_0^{2} + \frac{2}{2} \underbrace{\dot{\underline{r}}_0 \cdot \sum_{i=1}^{n} m_i \left(\underline{\omega} \times \underline{d}^{(i)}\right)}_{(\underline{r}_0 \times \omega) \cdot \underbrace{\sum_{i=1}^{n} m_i \underline{d}^{(i)}}_{=0}} + \underbrace{\frac{1}{2} \sum_{i=1}^{N} m_i \left(\underline{\omega} \times \underline{d}^{(i)}\right)^2}_{\text{Rotationsenergie}}$$

Schwerpunktsenergie

$=0$ weil Schwerpunkt im Ursprung von S*

Nebenrechnung: (skip $^{(i)}$)

$$\left(\underline{\omega} \times \underline{d}\right)^2 = \omega^2 d^2 \sin^2\alpha = \omega^2 d^2 (1 - \cos^2\alpha)$$

$$= \omega^2 d^2 - \omega^2 d^2 \cos^2\alpha$$

$$= \omega^2 d^2 - (\underline{\omega} \cdot \underline{d})^2 = (\omega_1^2 + \omega_2^2 + \omega_3^2)(d_1^2 + d_2^2 + d_3^2) - (\omega_1 d_1 + \omega_2 d_2 + \omega_3 d_3)^2$$

$$= \sum_{j,k=1}^{3} \omega_j \left(d^2 \delta_{jk} - d_j d_k\right) \omega_k$$

$$\Rightarrow T = \frac{1}{2} M \dot{r}_0^{2} + \frac{1}{2} \sum_{j,k=1}^{3} \omega_j \underbrace{\sum_{i=1}^{N} m_i \left(d^{(i)^2} \delta_{jk} - d_j d_k\right)}_{\underline{\underline{\Theta}}_{jk}} \omega_k$$

$$= \frac{1}{2} M \dot{r}_0^{2} + \frac{1}{2} \sum_{j,k=1}^{3} \omega_j \underline{\underline{\Theta}}_{jk} \omega_k$$

Trägheitstensor

$$= \frac{1}{2} M \dot{r}_0^{2} + \frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}} \underline{\omega}$$

$$= T_{\text{Translation}} + T_{\text{Rotation}}$$

im S' System

(b) analog Massendichte (kontinuierlich):

$$T = \frac{1}{2}\int d^3r\, \rho(\underline{d})\left(\dot{\underline{r}}_s + \underline{\omega}\times\underline{d}\right)^2$$

$$= \frac{1}{2}\underbrace{\int d^3r\, \rho(\underline{d})}_{M}\, \dot{r}_s^2 + (\dot{\underline{r}}_s \times \underline{\omega})\cdot \underbrace{\int d^3r\, \rho(\underline{d})\, \underline{d}}_{=0} + \frac{1}{2}\sum_{j,k=1}^{3} \omega_j \Theta_{jk}\, \omega_k$$

mit $\Theta_{jk} = \int d^3r\, \rho(\underline{d})\left[d^2 \delta_{jk} - d_j d_k\right]$

$$\Rightarrow T = \frac{1}{2} M \dot{r}_s^2 + \frac{1}{2}\underline{\omega}^T \underline{\underline{\Theta}}\, \underline{\omega}$$

6.2 Eigenschaften des Trägheitstensors

$$\Theta_{jk} = \sum_{i=1}^{N} m_i \left(d^{(i)2}\delta_{jk} - d_j d_k\right)$$

$$\Rightarrow \underline{\underline{\Theta}} = \sum_{i=1}^{N} m_i \begin{pmatrix} d_2^{(i)2} + d_3^{(i)2} & -d_1^{(i)} d_2^{(i)} & -d_1^{(i)} d_3^{(i)} \\ -d_2^{(i)} d_1^{(i)} & d_1^{(i)2} + d_3^{(i)2} & -d_2^{(i)} d_3^{(i)} \\ -d_3^{(i)} d_1^{(i)} & -d_3^{(i)} d_2^{(i)} & d_1^{(i)2} + d_2^{(i)2} \end{pmatrix}$$

analog: $\underline{r} = (x_1, x_2, x_3)$

$$\underline{\underline{\Theta}} = \int d^3r\, \rho(\underline{r}) \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_2 x_1 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_3 x_1 & -x_3 x_2 & x_1^2 + x_2^2 \end{pmatrix}$$