6. Mechanics of a rigid body

idea: spatially extended system of

(a) N masses with fixed distances $M = \sum_{i=1}^{N} m_i$

(b) continuous density of mass $M = \int d^3r\, \rho(\underline{r})$

coordinates:

(i) labsystem (system of inertia) : S

(ii) body-fixed coordinatesystem : S^*

(iii) translational system of (i) with center of mass as origin : S'

position in S^*: $\underline{r}^* = \underline{d} = \underline{r} - \underline{r}_0$

$\uparrow$ position, $\uparrow$ center of mass

$\Rightarrow |\underline{d}| = \text{const}$, but orientation can change

- Euler angles: φ, ϑ, ψ

φ: rotation around $\underline{e}_3$ $\rightarrow \underline{\underline{R}}_3(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

ϑ: rotation around $\underline{e}_1^{(\varphi)}$ $\rightarrow \underline{\underline{R}}_1(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}$

ψ: rotation around $\underline{e}_3^{(\vartheta)}$ $\rightarrow \underline{\underline{R}}_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ combined/chained rotation: $\underline{\underline{R}}(\varphi, \vartheta, \psi) = \underline{\underline{R}}_3(\psi)\, \underline{\underline{R}}_1(\vartheta)\, \underline{\underline{R}}_3(\varphi)$

6.1 kinetic energy & inertia tensor

(a) $T = \underbrace{\frac{1}{2} M \dot{\underline{r}}_0^2}_{\text{center of mass}} + \underbrace{\frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}\, \underline{\omega}}_{\text{rotation}}$

$\Theta_{jk} = \sum_{i=1}^{N} m_i \left[(d^{(i)})^2 \delta_{jk} - d_j d_k \right]$

(b) $T = \frac{1}{2} M \dot{\underline{r}}_0^2 + \frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}\, \underline{\omega}$

$\Theta_{jk} = \int d^3r\, \rho(\underline{d}) \left[d^2 \delta_{jk} - d_j d_k \right]$

Θ of sphere

$\rho(r) = \begin{cases} M/\frac{4}{3}\pi R^3 & r \leq R \\ 0 & r > R \end{cases}$

$\Theta = \Theta_1 = \Theta_2 = \Theta_3 = \frac{2}{5} M R^2$

$\underline{\underline{\Theta}} = \begin{pmatrix} \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \Theta \end{pmatrix}$

$\underline{\underline{\Theta}}$ ist ein Tensor 2. Stufe

↳ Transformation unter Drehungen:

Drehmatrix in $\mathbb{R}^3$ ($\underline{\underline{R}}^{-1}\underline{\underline{R}} = \underline{\underline{R}}^T\underline{\underline{R}} = 1$, $\det \underline{\underline{R}} = 1$)

Koordinaten: $x_\mu \to x_\mu' = \sum_{\nu=1}^{3} R_{\mu\nu} x_\nu$

$$\Rightarrow \quad \Theta_{\mu\nu} \to \Theta_{\mu\nu}' = \sum_{\lambda=1}^{3} \sum_{\sigma=1}^{3} R_{\mu\lambda} R_{\nu\sigma} \Theta_{\lambda\sigma}$$

Kompakt: $\underline{x} \to \underline{x}' = \underline{\underline{R}}\,\underline{x}$

$$\underline{\underline{\Theta}} \to \underline{\underline{\Theta}}' = \underline{\underline{R}}^T \underline{\underline{\Theta}}\,\underline{\underline{R}} \quad \text{(Ähnlichkeitstrafo)}$$

Bsp.: 1. Stufe („Vektor") $\quad A_\mu' = \sum_{\lambda=1}^{3} R_{\mu\lambda} A_\lambda$

2. Stufe („Matrix") $\quad A_{\mu\nu}' = \sum_{\lambda,\sigma=1}^{3} R_{\mu\lambda} R_{\nu\sigma} A_{\lambda\sigma}$

n. Stufe $\quad A'_{\underbrace{\mu\nu\ldots\xi}_{n\text{-mal}}} = \sum_{\lambda,\sigma,\ldots,\tau} R_{\mu\lambda} R_{\nu\sigma} \ldots R_{\xi\tau} A_{\lambda\sigma\ldots\tau}$

Trafonachweis: $\Theta_{\mu\nu} = \sum_{i=1}^{n} m_i \left[\left(\sum_{\tau=1}^{3} x_\tau^{(i)2} \right) \delta_{\mu\nu} - x_\mu^{(i)} x_\nu^{(i)} \right] = \ldots = \Theta_{\mu\nu}'$

mit $x_\mu' = \sum_{\lambda=1}^{3} R_{\mu\lambda} x_\lambda \Rightarrow$ (i) $\sum_{\tau=1}^{3} x_\tau'^2 = \sum_{\tau=1}^{3} \sum_{\lambda=1}^{3} \sum_{\sigma=1}^{3} R_{\tau\lambda} R_{\tau\sigma} x_\lambda x_\sigma$

$$= \sum_\lambda \sum_\sigma \underbrace{\sum_\tau R^T_{\lambda\tau} R_{\tau\sigma}}_{\delta_{\lambda\sigma}} x_\lambda x_\sigma$$

$$= \sum_{\lambda=1}^{3} x_\lambda^2 \quad \text{(invariant)}$$

(ii) $\sum_{\lambda,\sigma=1}^{3} R_{\mu\lambda} R_{\nu\sigma} \delta_{\lambda\sigma} = \sum_{\lambda=1}^{3} R_{\mu\lambda} R^T_{\lambda\nu} = \delta_{\mu\nu}$ (invariant)

$$\Rightarrow \sum_{\lambda,\sigma} R_{\mu\lambda} R_{\nu\sigma} \Theta_{\lambda\sigma} = \sum_{i=1}^{n} m_i \sum_\lambda \sum_\sigma R_{\mu\lambda} R_{\nu\sigma} \left(x^{(i)2} \delta_{\lambda\sigma} - x_\lambda^{(i)} x_\sigma^{(i)} \right)$$

$$= \sum_{i=1}^{n} m_i \left(x^{(i)\prime 2} \delta_{\mu\nu} - x_\mu^{(i)\prime} x_\nu^{(i)\prime} \right) = \Theta_{\mu\nu}'$$

Tensor in neuen Koordinaten

$\underline{\underline{\Theta}}$ ist diagonalisierbar durch eine orthogonale Transformation $\underline{\underline{R}} \in SO(3)$ (spezielle orthogonale Gruppe im $\mathbb{R}^3$)
↳ alle Drehungen

$$\underline{\underline{\Theta}}' = \overset{\underline{\underline{R}}^{-1}}{\underline{\underline{R}}^T}\, \underline{\underline{\Theta}}\, \underline{\underline{R}} = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix}$$

d.h. gedrehtes Koordinatensystem!) (y_1, y_2, y_3)

in Richtung der Hauptträgheitsachsen

$$\underline{\underline{\Theta}}' = \int d^3y\, \rho(\underline{y}) \begin{pmatrix} y_2^2 + y_3^2 & 0 & 0 \\ 0 & y_1^2 + y_3^2 & 0 \\ 0 & 0 & y_1^2 + y_2^2 \end{pmatrix}$$

$\Rightarrow \Theta_i \geq 0 \quad , i = 1,2,3$

Diagonalisierung $\Rightarrow$ Eigenwertproblem: $\underline{\underline{\Theta}}\, \underline{\vec{\omega}}^{(i)} = \Theta_i\, \underline{\vec{\omega}}^{(i)}$

Hauptachsenrichtungen (denn ist $\underline{\underline{\Theta}}$ diagonal)

$$\Leftrightarrow \det(\underline{\underline{\Theta}} - \Theta_i \mathbb{1}) = 0$$

$\Rightarrow$ 3 reelle, positive (semi-)definite Eigenwerte Θ_i

- $\underline{\underline{\Theta}}$ ist linear in $\rho(x)$: additiv beim Zusammenfügen 2er Körper
- Θ_i Hauptträgheitsmomente

$\Theta_1 \neq \Theta_2 \neq \Theta_3$ unsymmetrischer Kreisel

$\Theta_1 = \Theta_2 \neq \Theta_3$ symmetrischer Kreisel (Axialsymmetrie)

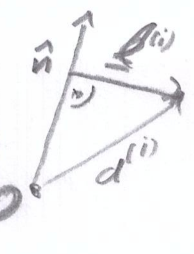
$\Theta_1 = \Theta_2 = \Theta_3$ "Kugelkreisel" (nicht notwendiger Weise Kugelform!)

- Trägheitsmoment

bzgl. Drehachse $\underline{\omega} = \omega \hat{\underline{n}}$

$$\Theta_n = \sum_{i,j=1}^{3} \hat{n}_i \Theta_{ij} \hat{n}_j = \hat{\underline{n}}^T \underline{\underline{\Theta}}\, \hat{\underline{n}} = \sum_{i=1}^{N} m_i \underbrace{\left[\underline{d}^{(i)2} - (\underline{d}^{(i)} \cdot \hat{\underline{n}})^2\right]}_{l^{(i)2} = \left(\underline{d}^{(i)} - \hat{\underline{n}}(\underline{d}^{(i)} \cdot \hat{\underline{n}})\right)^2}$$

$$= \sum_{i=1}^{N} m_i\, l^{(i)2} \quad \Rightarrow T_{\text{rot}} = \frac{1}{2} \Theta_n \omega^2$$



Satz von Steiner:

Sei $\Theta_{\mu\nu}$ der Trägheitstensor in einem im Schwerpunkt S zentrierten Körperfesten Sgstem S. Sei S' ein zu S achsenparalleles, um $\underline{a}$ verschobenes Sgstm. Dann gilt:

$$\Theta'_{\mu\nu} = \Theta_{\mu\nu} + M(a^2\delta_{\mu\nu} - a_\mu a_\nu)$$

(Trägheitsmoment bzgl $\hat{\underline{n}}$: $\Theta_n' = \Theta_n + Ml^2$, $l^2 = a^2 - (\underline{a}\cdot\hat{\underline{n}})^2$, $\Theta_n = \hat{\underline{n}}\,\underline{\underline{\Theta}}\,\hat{\underline{n}}$

Beweis: Transformation $\underline{\underline{\Theta}} \to \underline{\underline{\Theta}}'$ mit $\underline{r}' = \underline{r} + \underline{a}$

Bsp.: (i) Kugelsymmetrische Massendichte $\rho(\underline{r}) = \rho(r)$

$$\Rightarrow \Theta_1 = \Theta_2 = \Theta_3 =: \Theta$$

$$3\Theta = \sum_{i=1}^{3}\Theta_i = \int dx_1 dx_2 dx_3\, \rho(r)\Big[\underbrace{(x_2^2+x_3^2) + (x_3^2+x_1^2) + (x_1^2+x_2^2)}_{2r^2}\Big]$$

$$= 2\cdot 4\pi \int_0^R dr\, r^4 \rho(r)$$

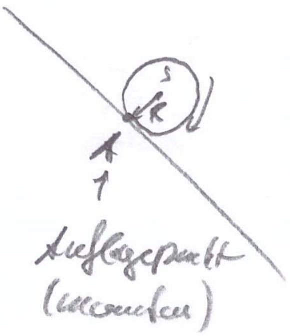
$$\Rightarrow \Theta = \frac{8\pi}{3}\int_0^R dr\, r^4\rho(r)$$

Für homogene Massenverteilung: $M = \underbrace{\frac{4\pi}{3}R^3}_{V}\rho(r)$:

$$\Theta = \frac{2}{R^3} M \int_0^R dr\, r^4 = \frac{2}{5}MR^2$$

(ii) Abrollende Kugel:

Θ bzgl. momentanem Drehachsendurch A:



$$\Theta_A = \Theta + MR^2$$

$$= \frac{2}{5}MR^2 + MR^2$$

$$= \frac{7}{5}MR^2$$

6.3 Drehimpuls & Bewegungsgleichungen

(a) $$\underline{L} = \sum_{i=1}^{N} m_i \underline{r}_i \times \underline{\dot r}_i = \sum_{i=1}^{N} m_i \left(\underline{r}_0 + \underline{d}^{(i)}\right) \times \left(\underline{\dot r}_0 + \underline{\omega} \times \underline{d}^{(i)}\right)$$

$$= \underbrace{\underbrace{\sum_{i=1}^{N} m_i}_{M} (\underline{r}_0 \times \underline{\dot r}_0)}_{= \underline{L}_S \text{ (Bahndrehimpuls)}} + \underbrace{\sum_{i=1}^{N} m_i \underline{d}^{(i)}}_{=0 \text{ Schwerpunkt}} \times \underline{\dot r}_0 + \underline{r}_0 \times \left(\underline{\omega} \times \underbrace{\sum_{i=1}^{N} m_i \underline{d}^{(i)}}_{=0}\right)$$

$$+ \underbrace{\sum_{i=1}^{N} m_i \underline{d}^{(i)} \times \left(\underline{\omega} \times \underline{d}^{(i)}\right)}_{\text{Relativdrehimpuls } \underline{L}_R \text{ (Eigendrehimpuls)}}$$

(b) $$\underline{L} = \underbrace{\underline{r}_0 \times M \underline{\dot r}_0}_{\underline{L}_S \text{ „Bahndrehimpuls" des Schwerpunkts}} + \underbrace{\int d^3r\, \rho(\underline{d})\, \underline{d} \times (\underline{\omega} \times \underline{d})}_{= \underline{L}_R}$$

$$= \underline{L}_R \underset{b(ac)-c(ab)}{=} \int d^3r\, \rho(\underline{d}) \left[d^2 \underline{\omega} - (\underline{d} \cdot \underline{\omega})\, \underline{d}\right]$$

$$\Rightarrow \underline{L}_R = \underline{\underline{\Theta}}\, \underline{\omega}$$

Komponentenweise: $$L_{R\mu} = \sum_{\nu=1}^{3} \underbrace{\int d^3r\, \rho(r) \left[d^2 \delta_{\mu\nu} - d_\mu d_\nu\right]}_{\Theta_{\mu\nu}} \omega_\nu$$

$\Rightarrow$ im Allgemeinen $\underline{L}_R \nparallel \underline{\omega}$, $\underline{\omega}$ kann von $\underline{\underline{\Theta}}$ gedreht werden!

$\underline{L}_R \parallel \underline{\omega}$, wenn $\underline{\omega}$ in Richtung einer Hauptträgheitsachse liegt.

$$\underline{\underline{\Theta}} = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix}$$