6.2 Properties of inertia tensor (continued)

(a) $\Theta_{jk} = \sum_{i=1}^{N} m_i \left[d^{(i)2} \delta_{jk} - d_j^{(i)} d_k^{(i)} \right]$

(b) $\Theta_{jk} = \int d^3r \; \rho(\underline{r}) \left(d^2 \delta_{ij} - d_j d_k \right)$

• diagonalization of $\underline{\underline{\Theta}}$: $\underline{x} \to \underline{x}' = \underline{\underline{R}} \, \underline{x}$

$$\underline{\underline{\Theta}} \to \underline{\underline{\Theta}}' = \underline{\underline{R}}^T \underline{\underline{\Theta}} \, \underline{\underline{R}}$$

such that $\underline{\underline{\Theta}}' = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix}$ $\quad \Theta_i$ principal moments of inertia

$\underline{\omega}^{(i)}$: principal axis

$\Rightarrow$ eigenvalue equation: $\underline{\underline{\Theta}} \, \underline{\vec{\omega}}^{(i)} = \Theta_i \underline{\omega}^{(i)}$

$\Theta_1 = \Theta_2 = \Theta_3$ spherical top (symmetry, not necessarily a sphere!)

$\Theta_1 = \Theta_2 \neq \Theta_3$ symmetric top

$\Theta_1 \neq \Theta_2 \neq \Theta_3$ asymmetric top

• moment of inertia w.r.t. $\underline{\omega} = \vec{\underline{n}} \, \omega$: $\Theta_n = \vec{\underline{n}}^T \underline{\underline{\Theta}} \, \vec{\underline{n}}$

$$= \sum_{i=1}^{N} m_i \left[d^{(i)2} - (\underline{d} \cdot \vec{\underline{n}})^2 \right]$$

$$= \sum_{i=1}^{N} m_i \left(\ell^{(i)} \right)^2$$

$\hat{=} \; \underline{d}^{(i)} - \vec{\underline{n}} (\underline{d}^{(i)} \cdot \vec{\underline{n}})$

$\Rightarrow T_{rot} = \frac{1}{2} \Theta_n \omega^2$

• Steiner's theorem: $\Theta'_{jk} = \Theta_{jk} + M \left(a^2 \delta_{jk} - a_j a_k \right)$

with rotation axes of S and S' shifted by $\underline{a}$

• $\Theta'_n = \Theta_n + M \ell^2$, $\ell^2 = a^2 - (\underline{a} \cdot \vec{\underline{n}})^2$, $\Theta_n = \vec{\underline{n}}^T \underline{\underline{\Theta}} \, \vec{\underline{n}}$

6.3 Angular momentum & equations of motion

(a) $$\underline{L} = \underbrace{M\,\underline{r}_0 \times \dot{\underline{r}}_0}_{L_s:\ \text{centre of mass}} + \underbrace{\sum_{i=1}^{N} m_i\, \underline{d}^{(i)} \times \left(\underline{\omega} \times \underline{d}^{(i)}\right)}_{L_R:\ \text{relative angular momentum}}$$

(b) $$\underline{L} = M\,\underline{r}_0 \times \dot{\underline{r}}_0 + \int d^3r\ \rho(\underline{d})\, \underline{d} \times (\underline{\omega} \times \underline{d})$$

$$\Rightarrow \underline{L}_R = \underline{\underline{\Theta}}\,\underline{\omega}$$

in principle: $\underline{L}_R \nparallel \underline{\omega}$ (rotation by $\underline{\underline{\Theta}}$)

wähle raumfestes System im Schwerpunkt: $\underline{r}_0 = 0$

$$\Rightarrow \underline{L} = \sum_{i=1}^{N} m_i\, \underline{d}^{(i)} \times (\underline{\omega} \times \underline{d}^{(i)}) = \underline{L}_R$$

$$\left(\text{Schwerpunkt: } \underline{r}_0 = \frac{1}{M} \sum_{i=1}^{N} m_i\, \underline{d}^{(i)} \right) \Rightarrow M \underline{r}_0 = \sum_{i=1}^{N} m_i\, \underline{d}^{(i)}$$

Newton'sche Gleichung für i-tes Teilchen: (vgl. 2.1)

$$m_i \ddot{\underline{d}}^{(i)} = \underbrace{\underline{F}^{(i)\,\text{außen}}}_{\text{äußere Kräfte z.B. } m_i \underline{g}} + \underbrace{\sum_{j=1}^{N} \underline{F}^{(ij)}}_{\text{innere Kräfte}}$$

$$\Rightarrow \underbrace{\sum_{i=1}^{N} m_i \ddot{\underline{d}}^{(i)}}_{M \ddot{\underline{r}}_0} = \sum_{i=1}^{N} \underline{F}^{(i)\,\text{außen}} + \sum_{i,j=1}^{N} \underline{F}^{(ij)}$$

mit actio = reactio $\left(\underline{F}^{(ij)} = -\underline{F}^{(ji)} \Rightarrow \underline{F}^{(ij)} + \underline{F}^{(ji)} = 0 \right)$ folgt:

Schwerpunktsatz: $$M \ddot{\underline{r}}_0 = \sum_{i=1}^{N} \underline{F}^{(i)\,\text{außen}} =: \underline{F}^{(\text{außen})}$$

Der Schwerpunkt bewegt sich so, als ob die gesamte Masse des starren Körpers in ihm vereint wäre und die **Resultierende** $\underline{F}^{(a)}$ aller äußeren Kräfte in ihm angreifen.

- innere Kräfte beeinflussen nicht die Schwerpunktsbewegung!

$$\underbrace{\sum_{i=1}^{N} m_i\, \underline{d}^{(i)} \times \ddot{\underline{d}}^{(i)}}_{\frac{d}{dt}\underline{L}} = \sum_{i=1}^{N} \underline{d}^{(i)} \times \underline{F}^{(i)\,\text{außen}} + \underbrace{\sum_{i,j=1}^{N} \underline{d}^{(i)} \times \underline{F}^{(ij)}}_{=0}$$

$$\underline{F}^{(ij)} = -\underline{F}^{(ji)}$$

- $\underline{d}^{(i)} \times \underline{F}^{(ij)} + \underline{d}^{(j)} \times \underline{F}^{(ji)} \overset{\downarrow}{=} (\underline{d}^{(i)} - \underline{d}^{(j)}) \times \underline{F}^{(ij)}$

$\underline{F}^{(ij)}$ entlang Verbindungsrichtung zwischen i und j $\rightarrow \underline{d}^{(i)} - \underline{d}^{(j)}$

$\Rightarrow$ **Drehimpulssatz:** $\sum_{i=1}^{N} m_i \underline{d}^{(i)} \times \ddot{\underline{d}}^{(i)} = \sum_{i=1}^{N} \underline{d}^{(i)} \times \underline{F}^{(i)\,\text{außen}}$

$\Leftrightarrow \quad \frac{d}{dt}\underline{L} = \underline{M}$ **Drehmoment**

(i) $$\frac{d}{dt}\underline{L} = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i\, \underline{d}^{(i)} \times (\underline{\omega} \times \underline{d}^{(i)})\right) = \frac{d}{dt}\left(L_i\, \underline{e}_i\right) = \frac{d}{dt}\sum_{j=1}^{N} \Theta_{ij}\, \omega_j\, \underline{e}_i$$

$$= \sum_{j=1}^{N} \Theta_{ij}\, \dot{\omega}_j\, \underline{e}_i + \sum_{j=1}^{N} \Theta_{ij}\, \omega_j\, \dot{\underline{e}}_i$$

$$= \underbrace{\sum_{j=1}^{N} \Theta_{ij}\, \dot{\omega}_j\, \underline{e}_i}_{\text{Ableitung für Beobachter im körperfesten System, } \underline{e}_i \text{ konstant}} + \sum_{j=1}^{N} \Theta_{ij}\, \omega_j\, (\underline{\omega} \times \underline{e}_i) = \frac{d^*}{dt}\underline{L} + \underline{\omega} \times \underline{L} = \underline{M}$$

$\frac{d^*}{dt}$: körperfeste Ableitung

(ii) alternativ: Transformation vom Laborsystem ($\underline{v}_0 = 0$) auf körperfestes System:

$$\frac{d}{dt} = \left(\frac{d}{dt}\right)^* + \omega \times$$

$$\Rightarrow \quad \frac{d}{dt}\underline{L} = \left(\frac{d}{dt}\right)^* \underline{L} + \omega \times \underline{L} = \underline{M}$$

$$= \underline{\underline{\Theta}}\, \dot{\underline{\omega}} + \underline{\omega} \times \underline{\underline{\Theta}}\, \underline{\omega} = \underline{M}$$

$\underline{\underline{\Theta}}$: fest im körperfesten System

$\Rightarrow$ **Eulersche Gleichungen:** körperfeste Achsen = Hauptträgheitsachsen: $L_i = \Theta_i \omega_i$

$$\Rightarrow \underline{\omega} \times \underline{\underline{\Theta}}\,\underline{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} \Theta_1 \omega_1 \\ \Theta_2 \omega_2 \\ \Theta_3 \omega_3 \end{pmatrix}$$

$$\Theta_1 \dot{\omega}_1 + (\Theta_3 - \Theta_2)\, \omega_2 \omega_3 = M_1$$

$$\Theta_2 \dot{\omega}_2 + (\Theta_1 - \Theta_3)\, \omega_1 \omega_3 = M_2$$

$$\Theta_3 \dot{\omega}_3 + (\Theta_2 - \Theta_1)\, \omega_2 \omega_1 = M_3$$

- nichtlinear, gekoppelt!

6.4. Beispiele

Bsp.: 1) Symmetrischer Kreisel: $\Theta_1 = \Theta_2 \equiv \Theta \neq \Theta_3$ (kräftefrei)

im Körperfesten System S^* (Sternchen im Folgenden weggelassen)

$$\Rightarrow \Theta_3 \dot{\omega}_3 + (\Theta - \Theta)\,\omega_2 \omega_1 = 0 \Rightarrow \omega_3 = \text{const}$$

$$\Theta_1 \dot{\omega}_1 = (\Theta - \Theta_3)\,\omega_2 \omega_3 \Rightarrow \dot{\omega}_1 = \frac{\Theta - \Theta_3}{\Theta}\,\omega_2 \omega_3$$

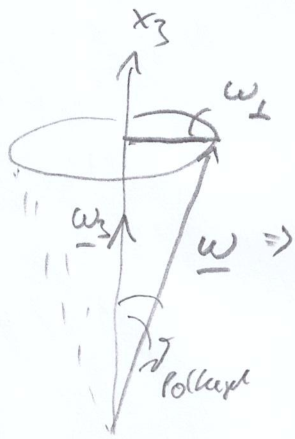
$$\Rightarrow \ddot{\omega}_1 = \frac{\Theta - \Theta_3}{\Theta}\,\omega_3 \dot{\omega}_2 \underset{\uparrow}{=} -\left(\frac{\Theta - \Theta_3}{\Theta}\,\omega_3^2\right)\omega_1$$

$$\dot{\omega}_2 = -\frac{\Theta - \Theta_3}{\Theta}\,\omega_1 \omega_3$$

$$\Rightarrow \omega_1(t) = \omega_\perp \sin(\omega_0 t - \varphi_0) \quad \text{mit} \quad \omega_0 = \frac{\Theta - \Theta_3}{\Theta}\,\omega_3$$

$$\omega_2(t) = -\omega_\perp \cos(\omega_0 t - \varphi_0)$$, $\omega_\perp, \varphi_0$ Integrationskonstanten

$$\Rightarrow \omega_1^2 + \omega_2^2 = \omega_\perp^2 = \text{const}$$



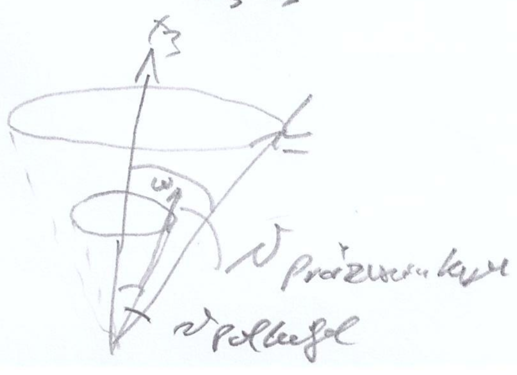
$\underline{\omega} \Rightarrow \omega_1^2 + \omega_2^2 + \omega_3^2 = \omega_\perp^2 + \omega_3^2 \Rightarrow \vartheta_{\text{Polkegel}} = \arctan \dfrac{\omega_\perp}{\omega_3}$

$\underline{\omega}$ und $\underline{L}$ (mit $L_j = \Theta_j \omega_j$) rotieren um Figurenachse

$$L^2 = \Theta^2 \omega_\perp^2 + \Theta_3^2 \omega_3^2 \quad (= L_\perp^2 + L_3^2)$$

$$\vartheta_{\text{Präzessionskegel}} = \arctan \frac{L_\perp}{L_3} = \arctan \frac{\Theta \omega_\perp}{\Theta_3 \omega_3} = \vartheta \text{ (Euler-Winkel)}$$

i.A. $\vartheta_{\text{Präzessionskegel}} \neq \vartheta_{\text{Polkegel}}$



Bislang: Körperfestes System: S^*

Nun: Raumfestes System: S

Euler-Winkel: $$\begin{pmatrix}\omega_1^*\\ \omega_2^*\\ \omega_3^*\end{pmatrix} = \dot\varphi\begin{pmatrix}\sin\psi\sin\vartheta\\ \cos\psi\sin\vartheta\\ \cos\vartheta\end{pmatrix} + \dot\vartheta\begin{pmatrix}\cos\psi\\ -\sin\psi\\ 0\end{pmatrix} + \dot\psi\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\left[\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \dot\varphi\begin{pmatrix}0\\0\\1\end{pmatrix} + \dot\vartheta\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} + \dot\psi\begin{pmatrix}\sin\varphi\sin\vartheta\\ -\cos\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}\right]$$

hier: $\dot\vartheta = 0 \Rightarrow$ $$\begin{pmatrix}\omega_1^*\\ \omega_2^*\\ \omega_3^*\end{pmatrix} = \dot\varphi\begin{pmatrix}\sin\psi\sin\vartheta\\ \cos\psi\sin\vartheta\\ \cos\vartheta\end{pmatrix} + \dot\psi\begin{pmatrix}0\\0\\1\end{pmatrix}$$

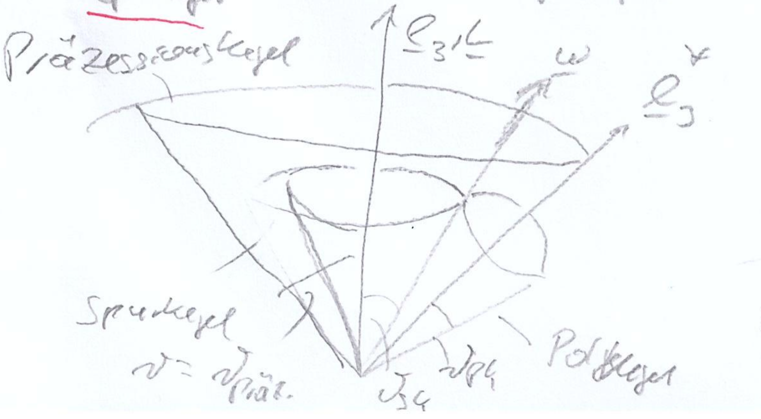
$$= \begin{pmatrix}\omega_\perp^*\sin(\Omega t)\\ \omega_\perp^*\cos(\Omega t)\\ \text{const}\end{pmatrix}$$

$\Rightarrow \psi = \Omega t \Rightarrow \dot\psi = \Omega = \text{const}$

$$\dot\varphi = \frac{\omega_\perp^*}{\sin\vartheta} = \text{const.} = \frac{\Theta_3\omega_3^*}{\Theta\cos\vartheta} = \Omega' = \frac{L}{\Theta}$$

$\Rightarrow$ Euler-Winkel: $\vartheta(t) = \vartheta_0$, $\psi(t) = \Omega t + \psi_0$, $\varphi(t) = \Omega' t + \varphi_0$

Spurkegel: $$\alpha_{\text{Spurkegel}} = \arctan\frac{\dot\psi\sin\vartheta}{\dot\varphi + \dot\psi\cos\vartheta} = \arctan\frac{\Omega\sin\vartheta}{\Omega' + \Omega\cos\vartheta}$$



wäre die Erde kräftefreier symm. Kreisel:

$$\frac{\Theta - \Theta_3}{\Theta} \approx \frac{1}{300}$$

in Realität: $\frac{1}{430}$ (Gezeiten....)