6.3 Angular momentum & equations of motion

Center of mass theorem: $M\ddot{\underline{r}}_0 = \sum_{i=1}^{N} \underline{F}^{(i)\,\text{external}} = \underline{F}^{\text{external}}$

Angular momentum theorem: $\sum_{i=1}^{N} m_i\, \underline{d}^{(i)} \times \ddot{\underline{d}}^{(i)} = \sum_{i=1}^{N} \underline{d}^{(i)} \times \underline{F}^{(i)\,\text{external}}$

$$\frac{d}{dt}\underline{L} = \underline{\underline{M}} \quad \text{(torque)}$$

- $\frac{d}{dt}\underline{L} = \frac{d^K}{dt^K}\underline{L} + \underline{\omega}\times\underline{L}$

  $\uparrow$ body-fixed derivative

- Euler's equations: $\frac{d}{dt}\underline{L} = \underline{\underline{\Theta}}\,\dot{\underline{\omega}} + \underline{\omega}\times\underline{\underline{\Theta}}\,\underline{\omega} = \underline{M}$

$$\underset{\substack{\uparrow \\ L_i = \Theta_i \omega_i}}{\Leftrightarrow} \begin{cases} \Theta_1 \dot{\omega}_1 + (\Theta_3 - \Theta_2)\,\omega_2\omega_3 = M_1 \\ \Theta_2 \dot{\omega}_2 + (\Theta_1 - \Theta_3)\,\omega_3\omega_1 = M_2 \\ \Theta_3 \dot{\omega}_3 = (\Theta_2 - \Theta_1)\,\omega_1\omega_2 = M_3 \end{cases}$$

coupled, nonlinear eqs.

6.4 Examples:

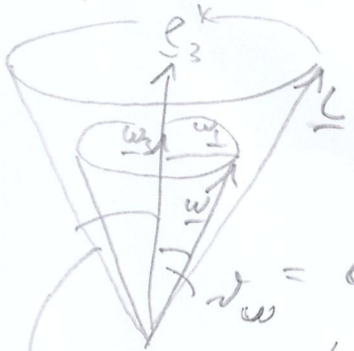
1) Torque-free top (symmetric: $\Theta_1 = \Theta_2 = \Theta \neq \Theta_3$)

$\Rightarrow \omega_3 = \text{const}$

$\omega_1(t) = \omega_\perp \sin(\omega_0 t - \varphi_0)$, $\quad \omega_0 = \frac{\Theta - \Theta_3}{\Theta}\,\omega_3$

$\omega_2(t) = -\omega_\perp \cos(\omega_0 t - \varphi_0)$

$\omega_1^2 + \omega_2^2 = \omega_\perp^2$



$\vartheta_\omega = \arctan\frac{\omega_\perp}{\omega_3}$ (Nile)

$\vartheta_{\text{precession}} = \arctan\frac{L_\perp}{L_3} = \arctan\frac{\Theta\omega_\perp}{\Theta_3\omega_3} = \vartheta$ (Euler angle)

System of inertia / lab frame:

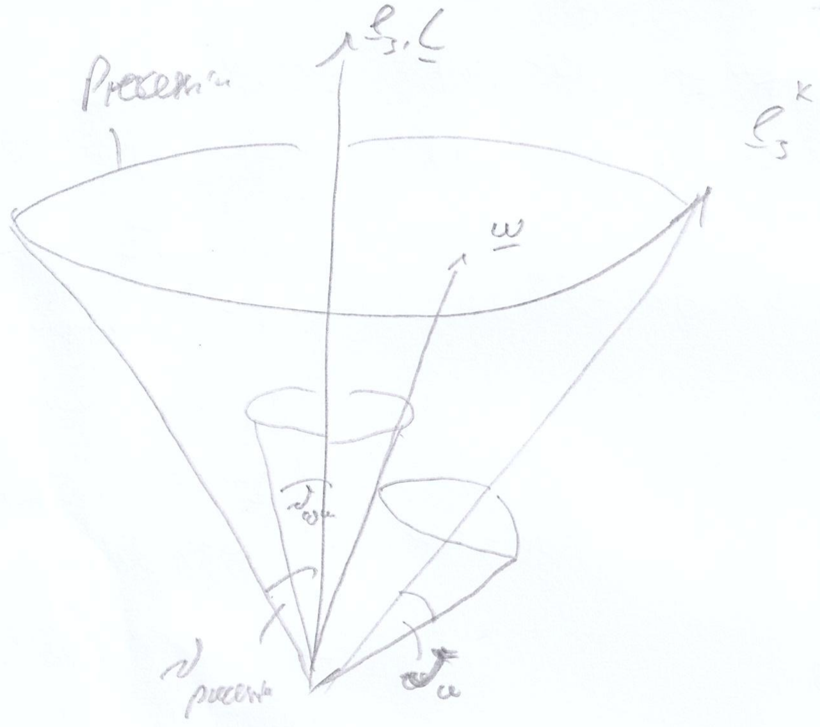
$$\begin{pmatrix} \omega_1^K \\ \omega_2^K \\ \omega_3^K \end{pmatrix} = \dot\varphi \begin{pmatrix} \sin\psi \sin\vartheta \\ -\cos\psi \sin\vartheta \\ \cos\vartheta \end{pmatrix} + \dot\psi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \quad \dot\vartheta = 0$$

$$= \begin{pmatrix} \omega_\perp^K \sin(\Omega t) \\ \omega_\perp^K \cos(\Omega t) \\ \text{const} \end{pmatrix}$$

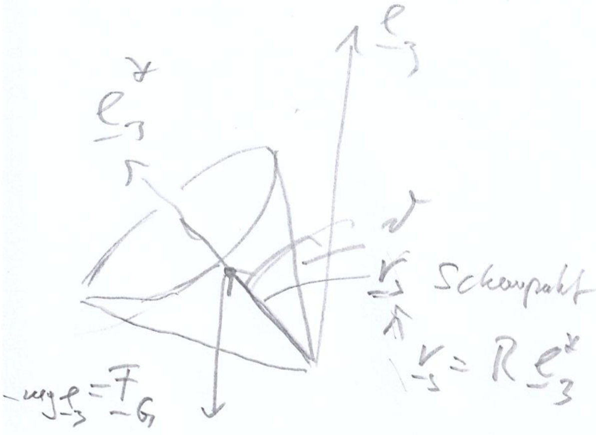
$\Rightarrow$ Euler angles: $\psi = \Omega t + \psi_0$, $\dot\varphi = \Omega' = \frac{\Theta_3 \omega_3^K}{\Theta \cos\vartheta}$, $\vartheta(t) = \vartheta_0$

$\dot\psi = \Omega \qquad \Rightarrow \varphi = \Omega' t + \varphi_0$

$\measuredangle(\underline{e}_3, \underline{e}_3^K) = \vartheta_{\text{cone } L} = \arctan \frac{\Omega \sin\vartheta}{\Omega' + \Omega\cos\vartheta}$



2) Schwerer, symmetrischer Kreisel:



$\underline{F}_G = -mg\,\underline{e}_3$ greift am Schwerpunkt an

Drehmoment: $\underline{M} = \underline{r}_S \times \underline{F}$

$$= -mgR\,\underline{e}_3^* \times \underline{e}_3$$

$\Rightarrow \underline{M} \perp$ Figurenachse

Gesamtenergie: $E = T + V = const$

$$V = mg\,\underline{e}_3 \cdot \underline{r}_S = mgR\,\underline{e}_3 \cdot \underline{e}_3^* = mgR\cos\vartheta$$

$$T = \frac{1}{2}\Theta(\omega_1^2 + \omega_2^2) + \frac{1}{2}\Theta_3\,\omega_3^2$$

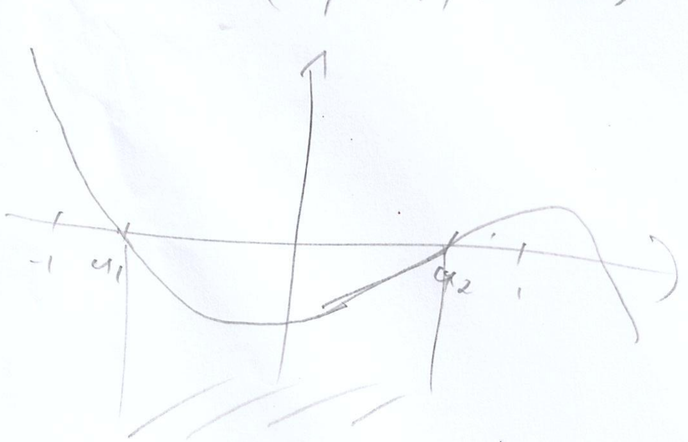
$$\Rightarrow E = \frac{\Theta}{2}\left(\dot\vartheta^2 + \dot\varphi^2\sin^2\vartheta\right) + \frac{\Theta_3}{2}\left(\dot\psi + \dot\varphi\cos\vartheta\right)^2 + mgR\cos\vartheta = const$$

$$\Leftrightarrow E = \frac{\Theta}{2}\dot\vartheta^2 + \frac{(L_3 - L_3^*\cos\vartheta)^2}{2\Theta\sin^2\vartheta} + \frac{L_3^{*2}}{2\Theta_3} + mgR\cos\vartheta$$

$$\underset{u=\cos\vartheta}{\Leftrightarrow} \quad \dot u^2 + V_{eff}(u) = 0, \quad \dot u = -\dot\vartheta\sin\vartheta$$

$$\Rightarrow V_{eff}(u) = \frac{2}{\Theta}\left(\frac{L_3^{*2}}{2\Theta_3} + mgRu - E\right)(1-u^2) + \left(\frac{L_3 - L_3^* u}{\Theta}\right)^2$$

$$= (C_1 + C_2 u)(1-u^2) + (C_3 - C_4 u)^2$$



Bewegung der Figurenachse: $-1 \le u_1 \le \cos\vartheta \le u_2 \le 1$ (Nutation)

$\Rightarrow$ Dynamisches System!

3) Kräftefreier unsymmetrischer Kreisel:

$0 < \Theta_1 < \Theta_2 < \Theta_3$

$$\dot\omega_1 = -\frac{\Theta_3-\Theta_2}{\Theta_1}\,\omega_2\omega_3 = -\alpha_1\omega_2\omega_3$$

$$\dot\omega_2 = \frac{\Theta_3-\Theta_1}{\Theta_2}\,\omega_3\omega_1 = \alpha_2\omega_3\omega_1$$

$$\dot\omega_3 = -\frac{\Theta_2-\Theta_1}{\Theta_3}\,\omega_1\omega_2 = -\alpha_3\omega_1\omega_2$$

Fixpunkte: $\underline{\omega}^* = \begin{pmatrix}\omega\\0\\0\end{pmatrix}$ oder $\begin{pmatrix}0\\\omega\\0\end{pmatrix}$ oder $\begin{pmatrix}0\\0\\\omega\end{pmatrix}$

(Rotation um x-   y-   z-Achse)

Linearisierung: $\delta\underline{\omega} = \underline{\omega} - \underline{\omega}^*$

$$\begin{pmatrix}\delta\dot\omega_1\\\delta\dot\omega_2\\\delta\dot\omega_3\end{pmatrix} = \underbrace{\begin{pmatrix}0 & -\alpha_1\omega_3 & -\alpha_1\omega_2\\ \alpha_2\omega_3 & 0 & \alpha_2\omega_1\\ -\alpha_3\omega_2 & -\alpha_3\omega_1 & 0\end{pmatrix}}_{=\underline{A}}\begin{pmatrix}\delta\omega_1\\\delta\omega_2\\\delta\omega_3\end{pmatrix}$$

Eigenwerte für:

$\omega^{*(1)} = \begin{pmatrix}\omega\\0\\0\end{pmatrix}$: $\quad 0 = \det(\underline{A} - \lambda\mathbb{1}) = \begin{vmatrix}-\lambda & 0 & 0\\ 0 & -\lambda & \alpha_2\omega\\ 0 & -\alpha_3\omega & -\lambda\end{vmatrix} = -\lambda(\lambda^2 + \alpha_2\alpha_3\omega^2)$

$\Rightarrow \lambda_1^{(1)} = 0, \quad \lambda_{2,3}^{(1)} = \pm i\omega\sqrt{\alpha_2\alpha_3}$ stabil / Zentrum

$\underline{\omega}^{*(2)} = \begin{pmatrix}0\\\omega\\0\end{pmatrix}$: $\quad 0 = \begin{vmatrix}-\lambda & 0 & -\alpha_1\omega\\ 0 & -\lambda & 0\\ -\alpha_3\omega & 0 & -\lambda\end{vmatrix} = -\lambda(\lambda^2 - \alpha_1\alpha_3\omega^2)$

$\Rightarrow \lambda_1^{(2)} = 0, \quad \lambda_{2,3}^{(2)} = \pm\omega\sqrt{\alpha_1\alpha_3}$ instabil (Sattelpunkt)

$\underline{\omega}^{*(3)} = \begin{pmatrix}0\\0\\\omega\end{pmatrix}$: $\quad 0 = \begin{vmatrix}-\lambda & -\alpha_1\omega & 0\\ \alpha_2\omega & -\lambda & 0\\ 0 & 0 & -\lambda\end{vmatrix} = -\lambda(\lambda^2 + \alpha_1\alpha_2\omega)$

$\Rightarrow \lambda_1^{(3)} = 0, \quad \lambda_{2,3}^{(3)} = \pm i\omega\sqrt{\alpha_1\alpha_2}$ stabil / Zentrum

$\Rightarrow$ Dynamische Systeme