7. Continuum mechanics

7.1 3 spring coupled oscillators

$$L = T - V$$
$$= \frac{m}{2} \sum_{i=0}^{N+1} \dot{q}_i^2 - \frac{D}{2} \sum_{i=0}^{N+1} (q_i - q_{i+1})$$

equation of motion: $m \ddot{q}_j - D(q_{j-1} - 2q_j + q_{j+1}) = 0 \quad j = 1,2,3.$

boundary conditions: $q_0(t) = 0 = q_{N+1}(t)$

solution / ansatz: $q_j(t) = C a_j \cos(\omega t - \delta)$

$$\hookrightarrow \begin{pmatrix} 2\frac{D}{m} & -\frac{D}{m} & 0 \\ -\frac{D}{m} & 2\frac{D}{m} & -\frac{D}{m} \\ 0 & -\frac{D}{m} & 2\frac{D}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$ eigenvalue equation

$$\Rightarrow \lambda_1 = \omega_1^2 = 2\frac{D}{m} \Rightarrow \omega_1 = \sqrt{2}\,\omega_0 \quad , \omega_0 = \frac{D}{m} \quad a^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_{2,3} = \omega_{2,3}^2 = (2 \pm \sqrt{2})\frac{D}{m} \Rightarrow \omega_2 = \sqrt{2 - \sqrt{2}}\,\omega_0 \quad , a^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\omega_3 = \sqrt{2 + \sqrt{2}}\,\omega_0 \quad , \underbrace{a^{(3)} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}}_{\text{on sine curves!}}$$

7.2 limit to continuum

$q_j(t) = C \sin(\alpha j) \cos(\omega t - \delta)$, $j = 1, \dots, N$, $q_0(t) = 0 = q_{N+1}(t)$

with $\alpha = \frac{\pi k}{N+1}$

$$\hookrightarrow \left[-m\omega^2 + 2D \left(1 - \cos\left(\frac{\pi k}{N+1}\right)\right) \right] C \sin(\alpha j) = 0$$

$$\hookrightarrow \omega_k = 2\sqrt{\frac{D}{m}} \sin\left(\frac{\pi k}{2(N+1)}\right)$$ checked for $k = 1,2,3$

$$\Rightarrow q_j(t) = \sum_{k=1}^{N} \sin\left(\frac{\pi k}{N+1} j\right) \underbrace{C_k \cos(\omega_k t - \delta_k)}_{[d_k \cos(\omega_k t) + e_k \sin(\omega_k t)]}$$

limits:

$N \to \infty \quad , \; l_0 \to 0 \; : \; (N+1)\, l_0 = l = \text{const}$ (total length)

$m \to 0 \quad , \; l_0 \to 0 \; : \; \frac{m}{l_0} = \rho = \text{const}$ (density of mass)

$D \to \infty \quad , \; l_0 \to 0 \; : \; D l_0 = \text{const}$ (total potential energy)

$= P = Y$ (module of elasticity)

with spatial coordinate: $x = j l_0 \to q_j(t) \to q(x,t)$

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} q(x,t) = \frac{\partial^2}{\partial x^2} q(x,t)$$ wave equation

$$\frac{1}{c^2} = \frac{m}{l_0} \frac{1}{D l_0}$$

$$\Rightarrow \quad a_j = a_{jk} = \sin\left(\frac{\pi k}{N+1} j\right) \qquad j,k = 1,\dots N$$

$$\Rightarrow \quad q_j(t) = \sum_{k=1}^{N} C_k \sin\left(\frac{\pi k}{N+1} j\right) \cos(\omega_k t - \delta_k)$$

$$= \sum_{k=1}^{N} \sin\left(\frac{\pi k}{N+1} j\right) \left(d_k \cos(\omega_k t) + e_k \sin(\omega_k t)\right)$$

Integrationskonstanten

$\Big| \cdot \sin\left(\frac{\pi s}{N+1} j\right)$, $\Big| \sum_{j=1}^{N}$

$$d_s = \frac{2}{N+1} \sum_{j=1}^{N} q_j(0) \sin\left(\frac{\pi s}{N+1} j\right), \quad e_s = \frac{2}{N+1} \frac{1}{\omega_s} \sum_{j=1}^{N} \dot{q}_j(0) \sin\left(\frac{\pi s}{N+1} j\right)$$

↳ Fourier-Zerlegung

Übergang zum Kontinuum:

$N \to \infty$ und $l_0 \to 0$: $(N+1) l_0 = l = \text{const}$ (Gesamtlänge)

$m \to 0$ und $l_0 \to 0$: $\frac{m}{l_0} = \rho = \text{const}$ (Massendichte)

$D \to \infty$ und $l_0 \to 0$: $D l_0 = \text{const} = Y = D$ (Potenzielle Energie) (Elastizitätsmodul)

Intuition:

7.3 Wellengleichung

$\Rightarrow$ Bewegungsgleichung:

$$\frac{m}{l_0} \frac{1}{D l_0} \ddot{q}_j = \frac{1}{l_0} \left[\frac{q_{j+1} - q_j}{l_0} - \frac{q_j - q_{j-1}}{l_0} \right]$$

Nun: Limes $l_0 \to 0$: $q_j(t) \to q(x,t)$ mit $x = j l_0$

$$\Rightarrow \frac{m}{l_0} \frac{1}{D l_0} \frac{\partial^2}{\partial t^2} q(x,t) = \lim_{l_0 \to 0} \frac{1}{l_0} \left[\frac{q(x+l_0,t) - q(x,t)}{l_0} - \frac{q(x,t) - q(x-l_0,t)}{l_0} \right]$$

$$= \lim_{l_0 \to 0} \frac{1}{l_0} \left(\left.\frac{\partial q}{\partial x}\right|_{x+\frac{l_0}{2}} - \left.\frac{\partial q}{\partial x}\right|_{x-\frac{l_0}{2}} \right)$$

Wellengleichung

$$= \frac{\partial^2 q}{\partial x^2} \Rightarrow \frac{1}{c^2} \frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\partial^2 q(x,t)}{\partial x^2} \quad \text{mit } \frac{1}{c^2} = \frac{m}{l_0} \frac{1}{D l_0}$$

mit $\frac{m}{l_0}\,\frac{1}{Dl_0} = \frac{\rho}{Dl_0} =: \frac{1}{c^2}$:

$$\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\psi(x,t) = \frac{\partial^2}{\partial x^2}\psi(x,t)$$ Wellengleichung

$$[c^2] = \left[\frac{Dl_0}{\rho}\right] = \left[\frac{\frac{F}{l_0}\,l_0}{\rho}\right]\left(= \left[\frac{F\,l_0}{m}\right]\right) = \frac{kg\,\frac{m}{s^2}}{\frac{kg}{m}} = \frac{m^2}{s^2} \Rightarrow [c] = \frac{m}{s}$$

c: Phasengeschwindigkeit

Lösungen:

(i) D'Alembertsche Lösung: $\psi(x,t) = f(x \pm ct)$

im mitbewegten Koordinatensystem: stehende Welle

(ii) Bernoullische Lösung: $\psi(x,t) = g(x)\,h(t)$

$$\Rightarrow \frac{1}{c^2}\,g(x)\,\ddot{h}(t) = g''(x)\,h(t) \Rightarrow \frac{1}{c^2}\,\frac{\ddot{h}(t)}{h(t)} = \frac{g''(x)}{g(x)} = -k^2 = \text{const.}$$

Also: $\ddot{h}(t) = -c^2k^2\,h(t)$ und $g''(x) = -k^2 g(x)$

$$\Rightarrow h(t) = h_1\cos(kct) + h_2\sin(kct) \quad \text{und} \quad g(x) = g_1\cos(kx) + g_2\sin(kx)$$

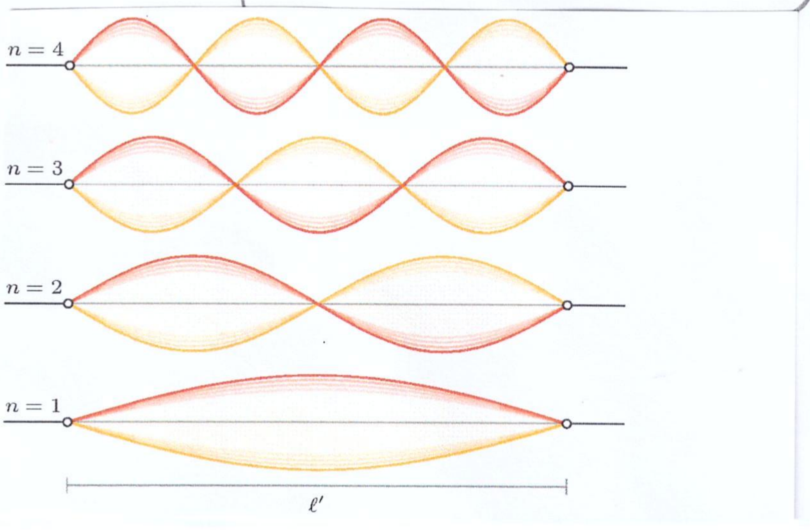
Randbedingung des festgehaltenen, schwingenden Mediums (unendliche Kette):

$$g(0) = g(\ell) = 0 \Rightarrow g_1 = 0,\ \sin(k\ell) = 0 \Rightarrow k\ell = n\pi \qquad n = 1, 2, \dots$$

$k_n = \frac{n\pi}{\ell}$

mit $k_n c t = \omega_n t$: Eigenfrequenzen $\quad \omega_n = n\frac{\pi c}{\ell}$

$$\Rightarrow \psi_n(x,t) = \sin\frac{n\pi x}{\ell}\left(d_n\cos(\omega_n t) + e_n\sin(\omega_n t)\right)$$ Eigenschwingung



4. Harmonische

3. Harmonische

2. Harmonische

Grundschwingung

$\Rightarrow$ allgemeine Lösung:

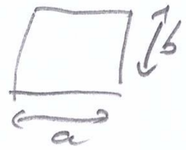
$$d_n = \frac{2}{l}\int_0^l \psi(x,0)\sin\frac{n\pi x}{l}\,dx \qquad e_n = \frac{2}{l}\frac{1}{\omega_n}\int_0^l \dot\psi(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$\psi(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(d_n\cos(\omega_n t) + e_n \sin(\omega_n t)\right)$$

Bsp.: Membran Schwingungen

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \qquad \text{mit } \psi = \psi(x,y,t)$$

Randbedingung einer rechteckigen Membran



$\hookrightarrow x=0,\ x=a,\ y=0,\ y=b$

$$\psi(0,y,t) = \psi(a,y,t) = \psi(x,0,t) = \psi(x,b,t) = 0 \quad \forall t \text{ (fest eingespannt)}$$

Produktansatz: $\psi(x,y,t) = \sin(kx)\sin(\hat k y)\left[d\cos(\omega t) + e\sin(\omega t)\right]$

mit $k = \frac{n\pi}{a}$, $\hat k = \frac{s\pi}{b}$, $n,s = 1,2,\dots$

$\hookrightarrow$ Eigenfrequenzen: $\omega_{ns} = \pi c\sqrt{\frac{n^2}{a^2} + \frac{s^2}{b^2}}$ irrationale Vielfache von $\omega_{11} = \pi c\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

allgemeine Lösung:

$$\psi(x,y,t) = \sum_{n,s=1}^{\infty} \sin\frac{n\pi x}{a}\sin\frac{s\pi y}{b}\left[d_{ns}\cos(\omega_{ns}t) + e_{ns}\sin(\omega_{ns}t)\right]$$

mit Fourier Koeffizienten:

$$d_{ns} = \frac{4}{ab}\int_0^a\int_0^b \psi(x,y,0)\sin\frac{n\pi x}{a}\sin\frac{s\pi y}{b}\,dx\,dy$$

$$e_{ns} = \frac{4}{ab}\frac{1}{\omega_{ns}}\int_0^a\int_0^b \dot\psi(x,y,0)\sin\frac{n\pi x}{a}\sin\frac{s\pi y}{b}\,dx\,dy$$

Frequenzen (i) i.A. irrationale Vielfache $\Rightarrow$ keine harmonischen Obertöne

etwa: bei $a=b$: $\omega_{12} = \pi c\sqrt{\frac{5}{a^2}} = \sqrt{2{,}5}\,\omega_{11}$

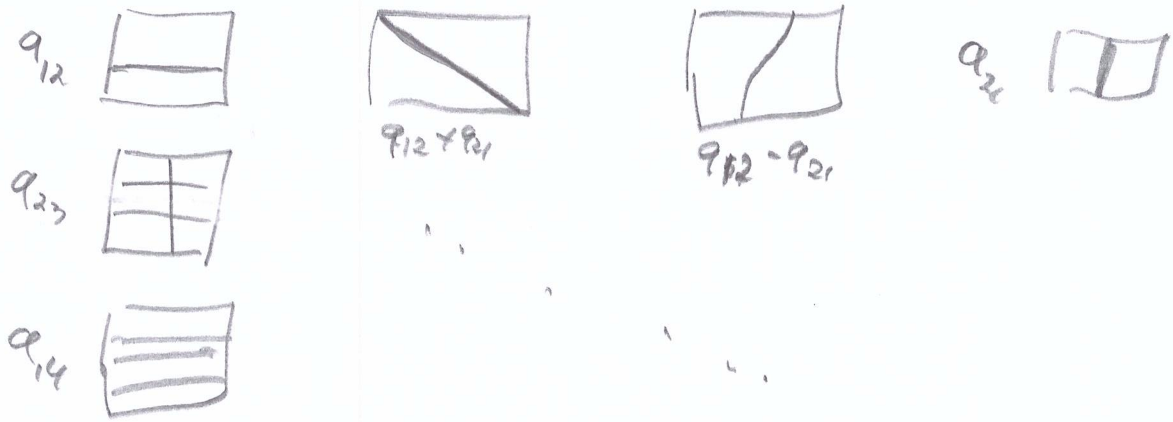
(ii) Entartung: Schwingungen mit gleicher Eigenfrequenz $\omega_{n_1 s_1} = \omega_{n_2 s_2}$:

$$\frac{n_1^2}{a^2} + \frac{s_1^2}{b^2} = \frac{n_2^2}{a^2} + \frac{s_2^2}{b^2} \Leftrightarrow \frac{a^2}{b^2} = \frac{n_1^2 - n_2^2}{s_2^2 - s_1^2} \Rightarrow \text{Bed. } \frac{a^2}{b^2} \in \mathbb{Q}$$

Knotenlinien: $\psi(x,y,t) \equiv 0$

(a) parallel zum Rand: $x = 0, \frac{a}{n}, \frac{2a}{n}, \dots, a$

$$y = 0, \frac{b}{s}, \frac{2b}{s}, \dots, b$$



(b) $a = b$, $d_{12}/d_{21} = e_{12}/e_{21} = u$, $\omega_{12} = \omega_{21} = \hat{\omega}$, Anregung nur der Eigenschwingungen q_{12}, q_{21}

$$\psi(x,y,t) = \left[u \sin\frac{\pi x}{a} \sin\frac{2\pi y}{a} + \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b}\right]$$

$$\left(d_{21}\cos(\hat{\omega} t) + e_{21}\sin(\hat{\omega} t)\right)$$

$$\Rightarrow [\dots] = 0 \underset{\sin(2z) = 2\sin z \cos z}{\Leftrightarrow} u\cos\frac{\pi y}{a} + \cos\frac{\pi x}{a} = 0$$

$$\left(\begin{aligned} &u \sin\frac{\pi x}{a}\, 2 \sin\frac{\pi y}{a}\cos\frac{\pi y}{a} + 2\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a} \\ &= 2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\left(u\cos\frac{\pi y}{a} + \cos\frac{\pi x}{a}\right)\end{aligned}\right)$$

$u = \pm 1$: Diagonale

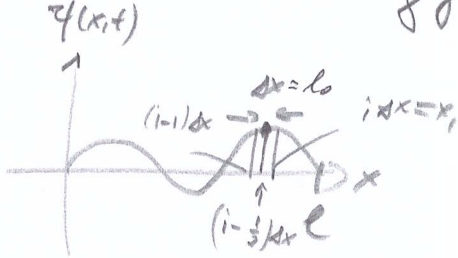
(c) Kreisförmige Membran: $\to$ Polarkoordinaten (r, φ)

~~Besselsche~~ Gleichung : $f(r,\varphi) = R(r)\,\phi(\varphi)$

$\hookrightarrow \phi'' + \lambda^2 \phi = 0$ Winkelanteil

$r^2 R'' + r R' + (k^2 r^2 - \lambda^2) R = 0$ Radialanteil (Bessel-DGL)

Limes der Lagrange-Funktion:



$$T=\sum_{i=1}^{N}\frac{m_i}{2}\left(\frac{dq_i}{dt}\right)^2=\sum_{i=1}^{N}\frac{\rho\,\Delta x}{2}\,\dot q_i^{\,2}$$

m_i ≙ Masse bei $x_i=(i-\frac{1}{2})\,\Delta x$ $\quad i=1,\dots,N$, ρ : Massendichte $\frac{m}{\Delta x}=\frac{m}{l_0}$

$$T=\lim_{N\to\infty}\sum_{i=1}^{N}\frac{\rho\,\Delta x}{2}\,\dot q_i^{\,2}=\frac{\rho}{2}\lim_{N\to\infty}\sum_{i=1}^{N}\Delta x\left(\frac{\partial\psi(x_i,t)}{\partial t}\right)^2=\frac{\rho}{2}\int_0^l dx\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2$$

Für V: $\quad \lim_{\Delta x\to 0}\frac{q_{i+1}(t)-q_i(t)}{\Delta x}=\frac{\partial\psi(x_i,t)}{\partial x}$

$$\Rightarrow\quad V=\lim_{N\to\infty}\frac{D}{2}\sum_{i=1}^{N-1}(q_{i+1}-q_i)^2=\lim_{N\to\infty}\sum_{i=1}^{N-1}\frac{D(\overset{\Delta x=l_0}{\Delta x})^2}{2}\frac{\left(\psi(x_{i+1},t)-\psi(x_i,t)\right)^2}{(\Delta x)^2}$$

geometrisch: $\Delta s=\sqrt{(\Delta x)^2+\left(\psi(x_{i+1},t)-\psi(x_i,t)\right)^2}$

$\simeq\Delta x\left[1+\frac{\left(\psi(x_{i+1},t)-\psi(x_i,t)\right)^2}{2(\Delta x)^2}\right]$

Abstand $i\longleftrightarrow i+1$

Kraft proportional zu $\Delta s-\Delta x$

Vorspannung der Feder $\Rightarrow V$ — " — zu $(\Delta s-\Delta x)^2$

$P=D\Delta x=\text{const}$

$$=\frac{D\Delta x}{2}\int_0^l dx\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2$$

$P=D\Delta x=Dl_0$: Vorspannung

Elastizitätsmodul

$\sum_{i=1}^{N} l_0\to\int_0^l dx$

$$\Rightarrow\quad L=T-V=\int_0^l dx\,\underbrace{\left[\frac{\rho}{2}\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2-\frac{\Delta x D}{2}\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2\right]}_{\mathcal{L}(\dot\psi,\psi')}$$

Lagrange-Dichte

- Hamiltonsches Prinzip:

$$\delta S=\delta\int_{t_1}^{t_2}dt\int_0^l dx\,\mathcal{L}(\psi,\dot\psi,\psi',x,t)=0$$

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi+\frac{\partial\mathcal{L}}{\partial\dot\psi}\delta\dot\psi+\frac{\partial\mathcal{L}}{\partial\psi'}\delta\psi'$$

• Variation: vgl. 3.2 Hamiltonsches Wirkungsprinzip

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial q'} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Partielle DGL!
(vgl. $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0,\ j=1,\dots,N$)

mit: $\mathcal{L} = \left(\rho\,\dot{q}^2 - D\Delta x\, q'^2\right)/2$ folgt:

$$\rho\,\frac{\partial^2}{\partial t^2} q - D\,\frac{\partial^2 q}{\partial x^2} = 0$$

Wellengleichung (freie Saite)

$$\Leftrightarrow\ \underbrace{\frac{\rho}{D\Delta x}}_{\frac{1}{c^2}}\,\frac{\partial^2}{\partial t^2} q = \frac{\partial^2}{\partial x^2} q \qquad \left(+\, q\, f(x,t)\right)$$

$f(x,t)$: äußere Kraftdichte

(Energiebeitrag: $\Delta V = -q f\,\Delta x$)

etwa: $f = -\rho g$ $\left(\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial q}{\partial t}\right)^2 - \frac{D}{2}\left(\frac{\partial q}{\partial x}\right)^2 + f(x,t)\,q\right) = T - V$

kin. Energiedichte

Typ: Wellengleichung für Feld q (Feldgleichung)

• 2/3D:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} q(x,y,z,t) = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\Delta\,:\ \text{D'Alembert-Operator}} q(x,y,z,t)$$