7.3 Wave equation

- circular / annular membrane

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt} \quad \text{with} \quad u(r=l, \varphi, t) = 0 \text{ (boundary condition)}$$

protocol: separation of (i) t and (r,φ) dependence

(ii) r and φ dependence

$$\Rightarrow u(r,\varphi,t) = R(r)\,\phi(\varphi)\,T(t) \quad , \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

(i) $\ddot{T} + c^2 k^2 T = 0 \quad \Rightarrow T(t) = A\cos(\omega t) + B\cos(\omega t)$

(ii) $\frac{d^2\phi}{d\varphi^2} + \lambda^2 \phi = 0 \quad \Rightarrow \phi(\varphi) = C\cos(\lambda\varphi + \varphi_0)$ with $\lambda \in \mathbb{Z}$ (2π periodicity)

(iii) $r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (k^2r^2 - \lambda^2)R = 0$ Bessel equation ($x = kr$)

Ansatz: $R(x) = x^\beta \sum_{s=0}^{\infty} \alpha_s x^s$

$$\Rightarrow R(r) = \sum_{m=0}^{\infty} D_m J_m(kr) \quad \text{with } J_m \text{ Bessel function}$$

$$\Rightarrow u(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left[A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\right]\cos(m\varphi + \varphi_0)\, J_m(k_{mn}r)$$

with $k_{mn} = \frac{\mu_{mn}}{l}$, μ_{mn}: n-th root of J_m (boundary conditions)

- Lagrange density in 3D: $L = \int_V d^3r \left[\frac{\rho}{2}\, \dot{\underline{q}}(r,t)^2 - V(\underline{q}, \underline{\nabla}\circ\underline{q})\right]$

density of potential energy

$$V = V^{\text{internal}} + V^{\text{external}}$$

$V^{\text{external}} = -\underline{f}(\underline{r},t)\cdot\underline{q}(r,t)$

$$V^{\text{internal}} \approx V^{\text{internal}}(0) + \sum_{j=1}^{3}\frac{\partial V^{\text{internal}}}{\partial \eta_{ij}}\eta_{ij} + \frac{1}{2}\sum_{ijkl}^{3}\underbrace{\frac{\partial^2 V^{\text{internal}}}{\partial \eta_{ij}\partial \eta_{kl}}}_{C_{ijkl}}\eta_{ij}\eta_{kl}$$

C_{ijkl} elasticity tensor

$$\underline{\underline{\eta}} = \underline{\nabla}\circ\underline{q} = \begin{pmatrix} \partial_x q_x & \partial_x q_y & \partial_x q_z \\ \partial_y q_x & \partial_y q_y & \partial_y q_z \\ \partial_z q_x & \partial_z q_y & \partial_z q_z \end{pmatrix}$$

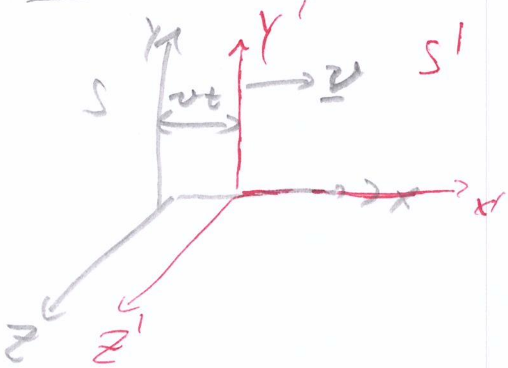
8. Spezielle Relativitätstheorie

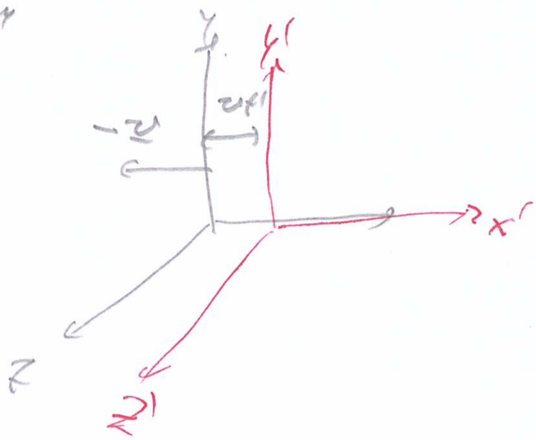
8.1 Postulate:

(i) Relativitätsprinzip: In allen zueinander gleichförmig bewegten Bezugssystemen (Inertialsystemen) laufen physikalische Vorgänge bei gleichen Bedingungen gleich ab

(ii) Konstanz der Lichtgeschwindigkeit: In allen Inertialsystemen ist die Lichtgeschwindigkeit im Vakuum gleich groß (unabhängig vom Bewegungszustand der Quelle)

8.2 Lorentz-Transformation





Bewegung in x-Richtung:

$x' = 0 \Leftrightarrow x = vt$ $\qquad$ $y' = 0 \Leftrightarrow y = 0$

$x = 0 \Leftrightarrow x' = -vt'$ $\qquad$ $z' = 0 \Leftrightarrow z = 0$

↳ lineare Trafo: $x' = A(x - vt)$, $\quad t' = Bt - Dx$ $\quad \Rightarrow B = A$

Kugelwelle mit Lichtgeschwindigkeit: $x^2 + y^2 + z^2 = c^2 t^2$ und $x'^2 + y'^2 + z'^2 = c^2 t'^2$ (Konstanz von c + Relativitätsprinzip)

$$\Rightarrow A^2(x^2 - 2xvt + v^2t^2) + y^2 + z^2 = c^2(A^2t^2 - 2AD\,tx + D^2x^2)$$

Vergleiche der Terme: x^2 nach links, t^2 nach rechts

$$x^2: \quad 1 = A^2 - c^2 D^2$$

$$xt: \quad 0 = -2A^2 v + 2c^2 AD \Rightarrow D = \frac{v}{c^2} A$$

$$t^2: \quad c^2 = c^2 A^2 - A^2 v^2$$

$$\Leftrightarrow 1 = A^2\left(1 - \frac{v^2}{c^2}\right) \Rightarrow A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- $c \to \infty \Rightarrow A = 1, \; D = 0 \Leftrightarrow$ Galilei-Trafo (vgl. 1.3)
  $t' = t$
  $x' = x - vt$

mit $v = \beta c$ ($\beta = \frac{v}{c}$) und $\gamma = \frac{1}{\sqrt{1-\beta^2}}$: $A = \gamma$, $cD = \beta\gamma$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\underline{\underline{\Lambda}}(v) \;\to\; \det(\underline{\underline{\Lambda}}(v)) = 1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Inverse:
$v \to -v$
$\beta \to -\beta$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Für beliebig orientierte Geschwindigkeiten $\underline{v}$: $s^2 = c^2t^2 - x^2 - y^2 - z^2$
$= \text{const}$

Addition 2 Trafos

$$\begin{tikzcd} S \arrow[r, "\underline{\underline{\Lambda}}(\underline{v})"] \arrow[rr, bend right, "\underline{\underline{\Lambda}}(\underline{w})"'] & S' \arrow[r, "\underline{\underline{\Lambda}}(\underline{w}')"] & S'' \end{tikzcd}$$

$$\Rightarrow \underline{\underline{\Lambda}}(\underline{w}')\,\underline{\underline{\Lambda}}(\underline{v}) = \underline{\underline{\Lambda}}(\underline{w})$$

$$\begin{pmatrix} \gamma_{w'} & -\beta_{w'}\gamma_{w'} & 0 & 0 \\ -\beta_{w'}\gamma_{w'} & \gamma_{w'} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_v & -\beta_v\gamma_v & 0 & 0 \\ -\beta_v\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{w'}\gamma_v(1+\beta_{w'}\beta_v) & -\gamma_{w'}\gamma_v(\beta_{w'}+\beta_v) & 0 & 0 \\ -\gamma_{w'}\gamma_v(\beta_{w'}+\beta_v) & \gamma_{w'}\gamma_v(1+\beta_{w'}\beta_v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\stackrel{!}{=} \begin{pmatrix} \gamma_w & -\beta_w\gamma_w & 0 & 0 \\ -\beta_w\gamma_w & \gamma_w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

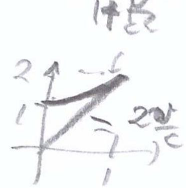
$$\Rightarrow \gamma_{w'}\gamma_v(1+\beta_{w'}\beta_v) = \gamma_w\,, \quad \gamma_{w'}\gamma_v(\beta_{w'}+\beta_v) = \beta_w\gamma_w$$

$$\Rightarrow \frac{\beta_{w'}+\beta_v}{1+\beta_{w'}\beta_v} = \beta_w \quad \Leftrightarrow \quad \frac{w'+v}{1+w'v/c} = w$$

- $\left.\begin{array}{l} v < c \\ w' < c \end{array}\right\} \Rightarrow w < c$

Steigung: $w' + v < 2c$

Steigung $1 + \frac{w'v}{c^2}$

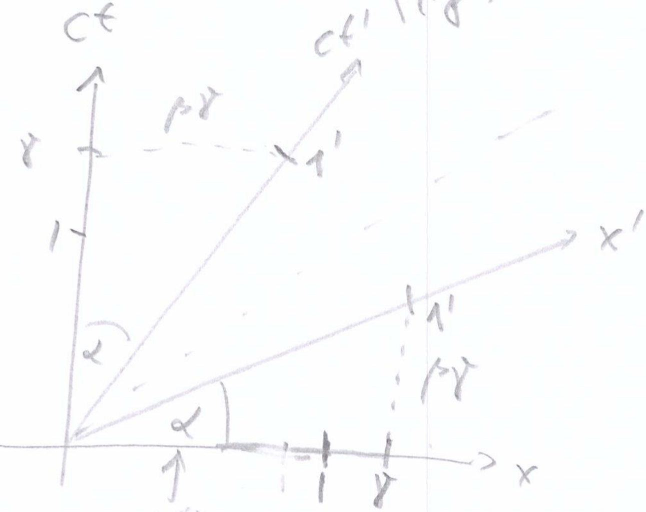


- $\left.\begin{array}{l} v = c \\ w' = c \end{array}\right\} \Rightarrow w = \dfrac{c+c}{1+1} = c$

8.3 Minkowski-Diagramm

2D: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \quad \Big| \quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$

$$= \begin{pmatrix} \gamma \\ \beta\gamma \end{pmatrix} ct' + \begin{pmatrix} \beta\gamma \\ \gamma \end{pmatrix} x'$$



$\beta\gamma = \frac{v}{c}\gamma < \gamma$

$\tan(\alpha) = \beta = \frac{v}{c}$

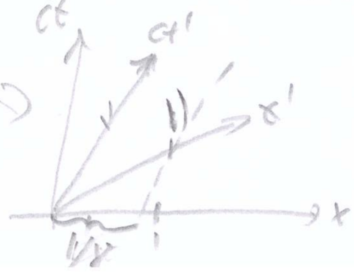
(i) Längenkontraktion: $c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$, $c\Delta t = \gamma(c\Delta t' + \beta\Delta x')$

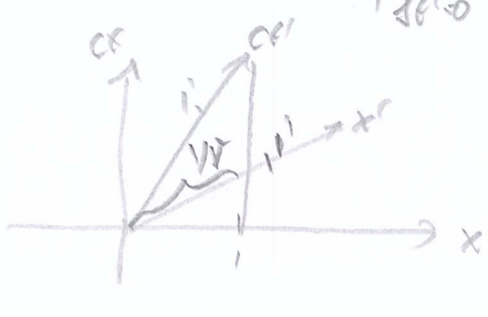
$\Delta x' = \gamma(\Delta x - \beta c\Delta t)$, $\Delta x = \gamma(\Delta x' + \beta c \Delta t')$

L_0 in S' in Ruhe: (bel. $\Delta t'$) $\Delta x' = L_0 \Rightarrow$ in S $L = \Delta x\big|_{\Delta t = 0} = \frac{\Delta x'}{\gamma} = \frac{L_0}{\gamma} < L_0$

$\Delta t = 0$

L_0 in S in Ruhe: (bel. Δt) $\Delta x = L_0 \Rightarrow$ in S' $L = \Delta x'\big|_{\Delta t' = 0} = \frac{\Delta x}{\gamma} = \frac{L_0}{\gamma} < L_0$





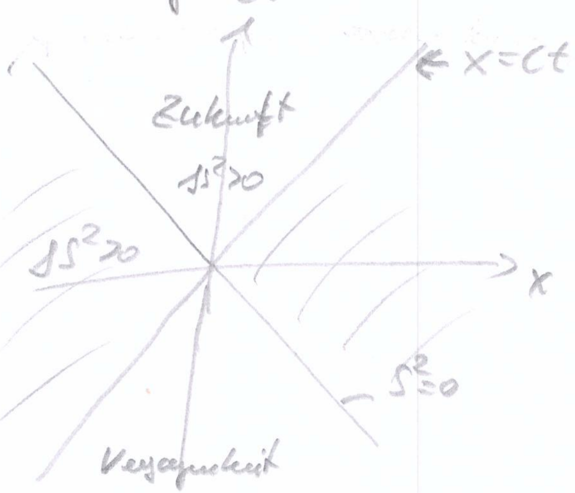
(ii) Zeitdilatation:

Uhr ruht in S': $\Delta T_0 = \Delta t' \Rightarrow$ in S: $\Delta T = \Delta t\big|_{\Delta x'=0} = \gamma \Delta t' = \gamma \Delta T_0 > \Delta T_0$

bewegte Uhren gehen langsamer

Uhr ruht in S: $\Delta T_0 = \Delta t \Rightarrow$ in S': $\Delta T = \Delta t'\big|_{\Delta x'=0} = \gamma \Delta t = \gamma \Delta T_0 > \Delta T_0$

(iii) Lichtkegel



$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2$$

8.4 Anwendungen:

(i) Lebensdauerverlängerung von Myonen

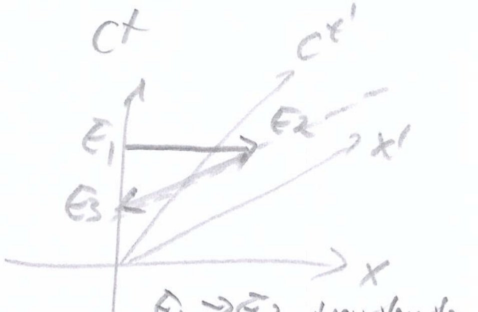
(ii) Leiter Paradoxon

(iii) Überlichtgeschwindigkeit ←

(iv) GPS

(v) Zwillingsparadoxon

(vi) 4er-Formalismus



$E_1 \rightarrow E_2$ instantan

$E_2 \rightarrow E_3$ — " —

- relativistische Mechanik : Zeitdilatation

$$\underline{F} = \frac{d\underline{p}}{dt} , \quad \text{hier: } t' = \gamma t \Rightarrow \frac{dt'}{dt} = \gamma(v)$$

mit v bewegt

$$\frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \leq 1$$

$$\text{mit } \quad v' = \gamma(v) v \quad \Big(\underline{p} = \underbrace{m\,\gamma(v)}_{=:m(v)} \underline{v} \Big)$$

mit Vierergeschwindigkeit: $\underline{u} = \frac{dx}{dt'} = \frac{dx}{dt'}\frac{dt'}{dt}$

$$\underline{u} = \gamma(v)\begin{pmatrix} c \\ \underline{v} \end{pmatrix} \Rightarrow u^2 = c^2$$

$\Rightarrow$ Viererbeschleunigung: $\underline{a} = \frac{du}{dt'} = \frac{d^2x}{dt'^2} = \gamma\frac{du}{dt} = \gamma\frac{d}{dt}\gamma(v)\begin{pmatrix} c \\ \underline{v} \end{pmatrix}$

$$\frac{d\gamma}{dt} = \frac{1}{c^2}\gamma^3\, \underline{v}\cdot\underline{a} \quad \left(\frac{d}{dt}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad + 2x \text{ Kettenregel} \right)$$

$$\Rightarrow \quad \underline{F} = m\gamma(v)\underline{a} + m\gamma^3(v)\frac{\underline{v}\cdot\underline{a}}{c^2} \neq$$

$\underline{F}$ als Viererkraft:

$$\Rightarrow \quad 0 = \gamma(cF^0 - \gamma\underline{F}\cdot\underline{v}) \Rightarrow \quad F^0 = \frac{\gamma}{c}\underline{F}\cdot\underline{v} = \frac{\gamma}{c}\underline{F}\cdot\frac{d\underline{x}}{dt} = \frac{\gamma}{c}\frac{\underline{F}\cdot d\underline{x}}{dt}$$

$$= \frac{\gamma}{c}\frac{dW}{dt} = \frac{d}{dt'}\frac{W}{c}$$

$$\text{mit } \quad p^0 = \gamma m c: \quad \frac{d}{dt'}W = \frac{d}{dt'}(c\,p^0) = \frac{d}{dt'}\left(\gamma(v) m c^2\right)$$

$$\Rightarrow \quad (p^0)^2 - \underline{p}^2 = \frac{E^2}{c^2} - \underline{p}^2 = m^2c^2$$

$$\Rightarrow \quad E = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} = mc^2\Big(1 + \underbrace{\frac{1}{2}\frac{p^2}{m^2c^2}}_{\text{nichtrelativistischer Term!}} + \dots\Big)$$