

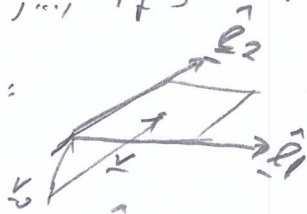
2.4 Generalisierte Koordinaten

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Idee: Finde $f = 3N - 1$ Koordinaten, die unabhängig voneinander sind und dennoch die Zwangsbedingungen erfüllen. *Generalisierte*

Koordinaten $\{q_1, \dots, q_f\}$ $f = \#$ Freiheitsgrade (unfrei holonom ZB)

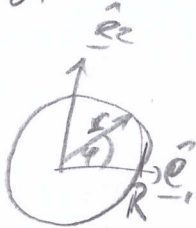
Bsp: (i) Ebene:



$$\underline{r}(t) = \underline{r}_0(t) + q_1 \underline{\hat{e}}_1 + q_2 \underline{\hat{e}}_2 \Rightarrow f=2$$

unabhängige Koordinatengitter

(ii) Kreis:



$$\underline{r}(t) = R (\cos \varphi \underline{\hat{e}}_1 + \sin \varphi \underline{\hat{e}}_2) \Rightarrow f=1$$

Ziel: Virtuelle Verschiebungen $\delta \underline{r}_i$ durch $\delta q_1, \dots, \delta q_f$ ausdrücken:

$$\underline{r}_i(q_1, \dots, q_f, t) \quad \delta \underline{r}_i = \sum_{j=1}^f \left(\frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j \Rightarrow \sum_{i=1}^N \underline{x}_i \cdot \delta \underline{r}_i = \sum_{j=1}^f \underbrace{\left(\sum_{i=1}^N \underline{x}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right)}_{Q_j} \delta q_j = 0$$

generalisierte Kräfte

2.5 Lagrange-Gleichungen 2. Art

$$\text{D'Alembertsches Prinzip: } \sum_{i=1}^N m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{i=1}^N \underline{x}_i \cdot \underline{f}_i = \sum_{j=1}^f Q_j \delta q_j$$

Kann sich durch Lösungsumformungen durch Definition der *Kinetischen*

$$\text{Energie } T = \sum_{i=1}^N \frac{1}{2} m_i v_i^2 \quad \text{zu: } \sum_{i=1}^N m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{j=1}^f \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad \text{mit } j=1, \dots, f$$

Lagrange-Gleichungen 2. Art

- gilt unbeeinträchtigt für Zwangsbedingungen (sofern keine generalisierten Koordinaten)

Für *konervative Kräfte*: $Q_j = -\frac{\partial V}{\partial q_j}$ mit $V(q_1, \dots, q_f, t)$

$$\Rightarrow \frac{d}{dt} \frac{\partial T-V}{\partial \dot{q}_j} - \frac{\partial T-V}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Lagrange-Funktion

2.4 Generalized coordinates

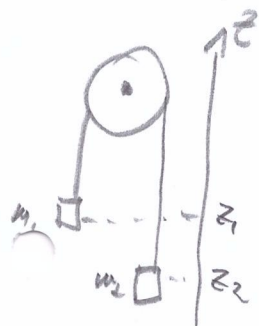
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set $\{q_1, \dots, q_f\}$ of independent coordinates that obey given constraints

idea: express virtual displacements $\delta \underline{r}_i$ ($i=1, \dots, N$) by δq_k ($k=1, \dots, f$)
 δx_j ($j=1, \dots, 3N$)

with $f = |\text{degrees of freedom}|$.

Case study Atwood's machine



(i) Newton $\underline{F} = m \underline{a}$

(ii) D'Alembert $\sum_{i=1}^N \underline{z}_i \cdot \delta \underline{r}_i = 0 = \sum_{i=1}^N (m_i \ddot{\underline{r}}_i - \underline{X}_i) \cdot \delta \underline{r}_i$

(iii) Lagrange I $m_j \ddot{x}_j - K_j - \sum_{\mu=1}^f \lambda_{\mu} \frac{\partial f_{\mu}}{\partial x_j} = 0$

$$\ddot{z}_1 = - \frac{m_1 - m_2}{m_1 + m_2} g$$

$$(iv) \text{ Lagrange II: } \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j=1, \dots, f$$

$$\text{hier } f=1 \Rightarrow \text{weil } z_1 + z_2 - l = \text{const} \Rightarrow z_1 = q, \quad z_2 = l - q$$

$$T = \frac{1}{2} m_1 \dot{z}_1^2 + m_2 \dot{z}_2^2 = \frac{1}{2} (m_1 + m_2) \dot{q}^2$$

$$V = m_1 g z_1 + m_2 g z_2 = g (m_1 q + m_2 (l - q)) = g (m_1 - m_2) q + m g l$$

$$\Rightarrow L = \frac{1}{2} (m_1 + m_2) \dot{q}^2 - g (m_1 - m_2) q - m g l$$

$$\Rightarrow \frac{d}{dt} [(m_1 + m_2) \dot{q}] + g (m_1 - m_2) = 0$$

$$\Rightarrow (m_1 + m_2) \ddot{q} + g (m_1 - m_2) = 0$$

$$\Rightarrow \ddot{q} = - \frac{m_1 - m_2}{m_1 + m_2} g$$

Einschub: Berechnung der linken Seite

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$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^f \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) \delta \mathbf{q}_j$$

$$= \sum_{j=1}^f \sum_{i=1}^N \left\{ \frac{d}{dt} m_i \dot{\mathbf{r}}_i \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) - m_i \dot{\mathbf{r}}_i \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) \right\} \delta \mathbf{q}_j$$

Mit $\frac{d}{dt} \mathbf{r}_i = \dot{\mathbf{r}}_i = \sum_{j=1}^f \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) \dot{\mathbf{q}}_j + \frac{\partial \mathbf{r}_i}{\partial t} = \dot{\mathbf{r}}_i$ folgt $\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) = \frac{\partial}{\partial \mathbf{q}_j} \dot{\mathbf{r}}_i$, und $\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\mathbf{q}}_j} = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j}$

damit: $\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) = \sum_k \left(\frac{\partial^2 \mathbf{r}_i}{\partial \mathbf{q}_k \partial \mathbf{q}_j} \right) \dot{\mathbf{q}}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial \mathbf{q}_j}$

$$\frac{\partial}{\partial \mathbf{q}_j} \dot{\mathbf{r}}_i = \frac{\partial}{\partial \mathbf{q}_j} \left\{ \sum_k \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_k} \right) \dot{\mathbf{q}}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right\} = \sum_k \left(\frac{\partial^2 \mathbf{r}_i}{\partial \mathbf{q}_j \partial \mathbf{q}_k} \right) \dot{\mathbf{q}}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial \mathbf{q}_j}$$

$$\Rightarrow \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_j \sum_i \left\{ \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) - m_i \dot{\mathbf{r}}_i \left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \mathbf{q}_j} \right) \right\} \delta \mathbf{q}_j \quad (*)$$

$= \frac{\partial}{\partial \mathbf{q}_j} \dot{\mathbf{r}}_i$

mit $m_i \dot{\mathbf{r}}_i \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) = \frac{\partial}{\partial \dot{\mathbf{q}}_j} \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right)$ und $m_i \dot{\mathbf{r}}_i \left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \mathbf{q}_j} \right) = \frac{\partial}{\partial \mathbf{q}_j} \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right)$ folgt:

$$(*)^2 \sum_{j=1}^f \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}_j} - \frac{\partial T}{\partial \mathbf{q}_j} - Q_j \right\} \delta \mathbf{q}_j = 0, \quad T = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$$

$\uparrow$
generalisierter Impuls

$$\Rightarrow \boxed{\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial T}{\partial \mathbf{q}_j} = Q_j}$$

Für konservative Kräfte: $Q_j = - \frac{\partial V}{\partial q_j}$

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$$(V(r_1(q_1, \dots, q_f, t), r_2(q_1, \dots, q_f, t), \dots, r_N(q_1, \dots, q_f, t))) = V(q_1, \dots, q_f, t)$$

$$\Rightarrow \frac{\partial}{\partial q_j} V = 0 \quad \text{im allgemeinen (s.o.)}$$

und die Definition der Lagrange-Funktion $L(\{q_j\}_{j=1}^f, \{\dot{q}_j\}_{j=1}^f, t) = T - V$ folgt:

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad j=1, \dots, f}$$

Anwandschema / Kochrezept:

Kochrezept der Lagrange-Gleichungen 2. Art:

1. Auswählen generalisierter Koordinaten: $q_1, \dots, q_f$, die die (holonomen) Zwangsbedingungen erfüllen: $\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_f, t) \equiv \mathbf{r}_i(q_k, t)$
2. Berechnen von Geschwindigkeiten $\mathbf{v}_i(q_k, \dot{q}_k, t) = \frac{d}{dt} \mathbf{r}_i(q_k, t)$ und kinetischer Energie $T(q_k, \dot{q}_k, t) = \frac{1}{2} \sum_{i=1}^N m_i \mathbf{v}_i(q_k, \dot{q}_k, t)^2$
3. Potenzielle Energie für konservative Kräfte: $V(q_k, t) = V(\mathbf{r}_i(q_k, t))$
 $\Rightarrow$ Lagrange-Funktion $L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k, t) - V(q_k, t)$
(Potenzielle/Lage-) Energie für nichtkonservative Kräfte: $Q_j = \sum_{i=1}^N \mathbf{X}_i \frac{\partial}{\partial q_j} \mathbf{r}_i(q_1, \dots, q_f, t)$
4. Aufstellung der Bewegungsgleichungen ($j = 1, \dots, f$):

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{bzw.} \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

5. Lösen der Bewegungsgleichungen

Bem.: Bei Geschwindigkeitsabhängigen Potenzial

- etwa ein elektromagnetisches Potenzial: $V_{em} = e(\phi - \mathbf{v} \cdot \mathbf{A})$ -

ergibt sich: $Q_j = - \frac{\partial V}{\partial q_j} + \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_j}$

Idee: $\sum_{i=1}^N \mathbf{X}_i \cdot \mathbf{F}_i = \sum_{i=1}^N \left(- \frac{\partial V}{\partial \mathbf{r}_i} + \frac{d}{dt} \frac{\partial V}{\partial \dot{\mathbf{r}}_i} \right) \mathbf{F}_i$

Für V_{em} : $Q_j = - \frac{\partial V_{em}}{\partial x_j} + \frac{d}{dt} \frac{\partial V_{em}}{\partial \dot{x}_j} = e \left(- \frac{\partial \phi}{\partial x_j} + \dot{\mathbf{r}} \cdot \frac{\partial \mathbf{A}}{\partial x_j} - \frac{dA_j}{dt} \right)$

$$= e \left(- \frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t} - \sum_{k=1}^3 \frac{\partial A_k}{\partial x_k} \dot{x}_k + \sum_{k=1}^3 \dot{x}_k \frac{\partial A_k}{\partial x_j} \right)$$

$$= e \left(- \frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t} + \sum_{k=1}^3 \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \right) \dot{x}_k \right) \Rightarrow \mathbf{F}_{el} = e \left(- \nabla \phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \right)$$

$$\text{Allgemein: } \sum_{i=1}^N \underline{x}_i \cdot \underline{\dot{r}}_i = \sum_{j=1}^f \left(\sum_{i=1}^N \underline{x}_i \cdot \frac{\partial \underline{r}_i}{\partial \underline{q}_j} \right) \delta \underline{q}_j$$

$$= \sum_{j=1}^f \left(\sum_{i=1}^N \left(-\frac{\partial V}{\partial \underline{r}_i} + \frac{d}{dt} \frac{\partial V}{\partial \dot{\underline{r}}_i} \right) \cdot \frac{\partial \underline{r}_i}{\partial \underline{q}_j} \right) \delta \underline{q}_j$$

$$= \sum_{j=1}^f \left(-\frac{\partial V}{\partial \underline{q}_j} + \sum_{i=1}^N \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{\underline{r}}_i} \right) \cdot \frac{\partial \underline{r}_i}{\partial \underline{q}_j} \right) \delta \underline{q}_j$$

$$\frac{\partial V}{\partial \underline{q}_j} = \sum_{i=1}^N \frac{\partial V}{\partial \underline{r}_i} \frac{\partial \underline{r}_i}{\partial \underline{q}_j} = \sum_{i=1}^N \frac{\partial V}{\partial \dot{\underline{r}}_i} \frac{\partial \dot{\underline{r}}_i}{\partial \underline{q}_j}$$

$$= \sum_{j=1}^f \left(-\frac{\partial V}{\partial \underline{q}_j} + \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{\underline{q}}_j} \right) \right) \delta \underline{q}_j$$

$$= \sum_{j=1}^f \underbrace{\left(-\frac{\partial V}{\partial \underline{q}_j} + \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{\underline{q}}_j} \right) \right)}_{Q_j} \delta \underline{q}_j$$

$$(i) \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = \sum_{i=1}^N m \ddot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \frac{d}{dt} \left(\sum_{i=1}^N \frac{m_i}{2} \dot{\vec{x}}_i^2 \right) = \frac{d}{dt} T$$

$$(ii) \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{\partial V_{ij}}{\partial \vec{r}_i} \right) + \frac{\partial V_i}{\partial \vec{r}_i} \right) \cdot \dot{\vec{r}}_i$$

Konservative Kräfte

$$= - \frac{d}{dt} \left(\frac{1}{2} \sum_{i,j=1}^N V_{ij}(\vec{r}_{ij}) + \sum_{i=1}^N V_i(\vec{r}_i) \right)$$

$$= - \frac{d}{dt} V$$

$$\left(\frac{d}{dt} V_{ij}(\vec{r}_{ij}) = \left(\frac{\partial V_{ij}}{\partial \vec{r}_{ij}} \right) \cdot \dot{\vec{r}}_i + \left(\frac{\partial V_{ij}}{\partial \vec{r}_{ij}} \right) \cdot \dot{\vec{r}}_j \right)$$

$$i+ii \Rightarrow \frac{dT}{dt} + \frac{dV}{dt} = 0$$

(evtl. können dissipative Prozesse, z.B. Reibung, die Energieerhaltung stören/aufheben)

$$\frac{d}{dt}(T+V) = \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i$$

Wird bei Berücksichtigung von Zwangsbedingungen: $\sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i$ auf rechte Seite

$$\frac{d}{dt}(T+V) = - \sum_{\mu=1}^M \lambda_{\mu} \frac{\partial f_{\mu}}{\partial t}$$

$$\text{wegen } \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = \sum_{\mu=1}^M \sum_{i=1}^N \lambda_{\mu} \left(\frac{\partial f_{\mu}}{\partial \vec{r}_i} \right) \cdot \dot{\vec{r}}_i$$

$$\text{und } 0 = \frac{df_{\mu}}{dt} = \sum_{i=1}^N \vec{v}_i \cdot \frac{\partial f_{\mu}}{\partial \vec{r}_i} + \frac{\partial f_{\mu}}{\partial t}$$

$\Rightarrow$ Energie ist erhalten, wenn Zwangsbedingungen skleronom sind. $\left(\frac{\partial f_{\mu}}{\partial t} = 0 \right)$

Lagrange II & Newton: (Konservative Kräfte) (1 Punktformale)

$$T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m \sum_{j=1}^3 \dot{x}_j^2, \quad q_1 = x_1, \quad q_2 = x_2, \quad q_3 = x_3$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left[\left(\frac{1}{2} m \sum_{k=1}^3 \dot{q}_k^2 \right) - V(q_1, q_2, q_3) \right] - \frac{\partial}{\partial q_j} [T(q_1, q_2, q_3) - V(q_1, q_2, q_3)] = \frac{d}{dt} m \dot{q}_j + \frac{\partial V}{\partial q_j} = 0$$

$$\Leftrightarrow m \ddot{q}_j = - \frac{\partial V}{\partial q_j} \quad j=1, 2, 3$$