2 D'Alembert's principle

2.1 Constraints and forces

System of N mass points → $3N$ degrees of freedom (without constraints)

idea: reduction of degrees of freedom by constraints.

(i) holonomic: $f_\mu(\underline{r}_1,\dots,\underline{r}_N,t)=0 \quad ,\mu=1,\dots,\Lambda \Rightarrow f=3N-\Lambda$

total differential: $df_\mu = \sum_{i=1}^{N} \nabla_{\underline{r}_i} f_\mu \cdot d\underline{r}_i + \frac{\partial f_\mu}{\partial t}\, dt = 0$

(ii) nonholonomic: $\sum_{i=1}^{N} \underline{a}_{\mu i}(\underline{r}_1,\dots,\underline{r}_N,t)\cdot d\underline{r}_i + \underline{a}_{\mu 0}(\underline{r}_1,\dots,\underline{r}_N,t)\, dt = 0$

but cannot be written as $df_\mu = \dots = 0$ (even with an integrating factor: $\nexists g_\mu : \underline{a}_{\mu i} = g_\mu \nabla_{\underline{r}_i} f_\mu$

(meaning: motion locally restricted)

(a) rheonomic: explicitly time dependent

(b) scleronomic: not explicitly time dependent

equations of motion $m_i \ddot{\underline{r}}_i = \underset{\uparrow \text{ external}}{\underline{F}_i} + \underbrace{\sum_{j=1}^{N} \underline{F}_{ij}}_{\text{internal forces}} =: \underline{X}_i \,, \quad i=1,\dots,N$

↳ solution given constraints.

Idea: turn constraints into forces governing the motion:

$$m_i \ddot{\underline{r}}_i = \underline{X}_i + \underline{Z}_i$$

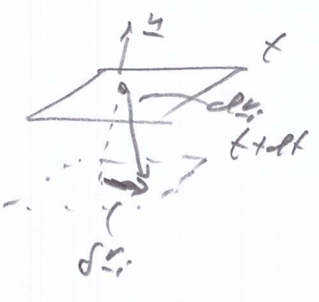
2.2 Virtual displacements

- (hypothetical) deviation from trajectory obeying the constraints (for fixed time)
- ex. motion on oscillating plane: $f = \underline{n}\cdot(\underline{r} - \underbrace{\underline{r}_0(t)}_{\text{motion of plane}}) = 0$

$\Rightarrow df = \underline{n}\cdot(d\underline{r}_i - \underline{v}_0(t)\,dt) = 0$ with $\underline{v}_0 = \frac{d}{dt}\underline{r}_0(t)$

$\Rightarrow \underline{n}\cdot d\underline{r}_i = \underline{v}_0(t)\,dt$ (most often not zero)

but $\delta f = \underline{n}\cdot\delta\underline{r}_i \underset{\substack{\uparrow \\ \text{fixed time}}}{=} 0 \Rightarrow \delta\underline{r}_i \perp \underline{n}$



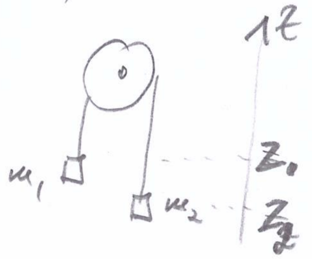
2.3 D'Alembert's principle

virtual work: $\sum_{i=1}^{N} \underline{Z}_i \cdot \delta \underline{r}_i = 0 \Rightarrow$ D'Alembert's principle: $\sum_{i=1}^{N} (m_i \ddot{\underline{r}}_i - \underline{X}_i) \cdot \delta \underline{r}_i = 0$

(note: no constraints explicitly included)

ex: (i) $\underline{Z}_i \cdot \delta \underline{r}_i = \lambda \underbrace{\underline{\nabla}_r f}_{=\underline{n}} \cdot \underbrace{\delta \underline{r}}_{\parallel \text{ to plane}} = 0$

(ii) Atwood's machine: m_1 & m_2 connected via a pulley by a string



$$z_1 + z_2 = \text{const} \Rightarrow \ddot{z}_1 = -\ddot{z}_2, \quad \delta z_1 = -\delta z_2$$

$$\Big(m_1 \ddot{z}_1 - \underbrace{(-m_1 g)}_{X_1}\Big)\delta z_1 + \Big(m_2 \ddot{z}_2 - \underbrace{(-m_2 g)}_{X_2}\Big)\delta z_2 = 0$$

$$\Rightarrow \underbrace{\left[(m_1+m_2)\ddot{z}_1 + (m_1 - m_2) g\right]}_{\stackrel{!}{=} 0} \delta z_1 = 0 \text{ for arbitrary } \delta z_1$$

$$\Rightarrow \ddot{z}_1 = -\frac{m_1 - m_2}{m_1 + m_2} g$$

general notation: $\underline{r}_i \to x_j$, $\underline{X}_i \to K_j$, $\underline{a}_{\mu i} \to \phi_j^{\mu}$, $\underline{\nabla}_i f_\mu \to \phi_j^\mu$

$i = 1, \dots, N$, $j = 1, \dots, 3N$

$$\Rightarrow \sum_{j=1}^{3N} (m_j \ddot{x}_j - K_j)\, \delta x_j = 0 \quad \text{given} \quad \sum_{j=1}^{3N} \phi_j^{\mu} \delta x_j = 0$$

idea: Lagrange multipliers (add constraints to equation of motion)

$$\hookrightarrow \sum_{j=1}^{3N} \Big(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu \phi_j^{\mu}\Big)\delta x_j = 0$$

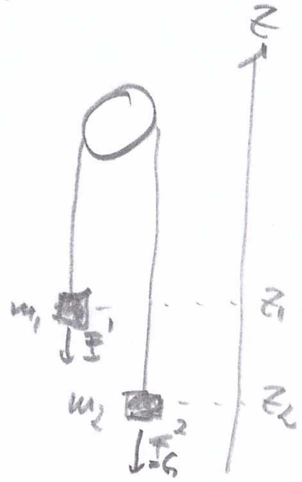
choose/determine $\lambda_1(t), \dots, \lambda_\Lambda(t)$ such that $m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu \phi_j^\mu = 0$, $j = 1, \dots, \Lambda$

insert into remaining equations: $\sum_{j=\Lambda+1}^{3N} \underbrace{\Big(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu(t) \phi_j^\mu\Big)}_{=0} \underbrace{\delta r_j}_{\text{free}} = 0$

$\Rightarrow$ Lagrange's equation of the first kind:

$$m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu(t) \phi_j^\mu = 0 \qquad j = 1, \dots, 3N$$

Einschub Beispiel: Atwoodsche Fallmaschine auf 4 Varianten (Beispiel)



(i) Newton: $\underline{F} = m\underline{a}$

$$F^{(1)} = -m_1 g + m_2 g = -(m_1 - m_2) g$$

$$F_{Ges} = m_{ges}\ddot{z} = (m_1+m_2)\ddot{z}$$

$$\Rightarrow \quad (m_1+m_2)\ddot{z} = -(m_1-m_2) g$$

$$\Leftrightarrow \quad \ddot{z} = -\frac{m_1-m_2}{m_1+m_2} g$$

(ii) D'Alembert $\sum_{i=1}^{N} \underline{Z}_i \cdot \delta \underline{r}_i = 0 = \sum_{i=1}^{N} (m_i \ddot{\underline{r}}_i - \underline{X}_i)\delta \underline{r}_i$

$$\delta z_1 = -\delta z_2, \quad \underline{Z}_1 = \underline{Z}_2 \Rightarrow \sum_{i=1}^{2} \underline{Z}_i \delta \underline{r}_i = Z(\delta z_1 + \delta z_2) = Z(\delta z_1 - \delta z_2) = 0$$

$$\left(m_1 \ddot{z}_1 - (-m_1 g)\right)\delta z_1 + \left(m_2 \underbrace{\ddot{z}_2}_{=-\ddot{z}_1} - (-m_2 g)\right)\underbrace{\delta z_2}_{=-\delta z_1} = 0$$

$$\Leftrightarrow \quad \underbrace{\left[(m_1+m_2)\ddot{z}_1 + (m_1-m_2) g\right]}_{\overset{!}{=}0}\delta z_1 = 0 \Rightarrow \ddot{z}_1 = -\frac{m_1-m_2}{m_1+m_2} g$$

(iii) Lagrange I: $y_1 = y_2 = x_1 = x_2 = 0$, $z_1 + z_2 = l = const$

$$m_j \ddot{x}_j - X_j - \sum_{\mu=1}^{\lambda} \lambda_\mu \frac{\partial f}{\partial x_j} = 0$$

$$\Rightarrow f = z_1 + z_2 - l = 0$$

$$\Rightarrow df = dz_1 + dz_2 = 0$$

$\Rightarrow$ I: $m_1 \ddot{z}_1 - (-m_1 g) - \lambda \cdot 1 = 0$ und $\dot{z}_1 + \dot{z}_2 = 0$, $\ddot{z}_1 = -\ddot{z}_2$

II: $m_2 \ddot{z}_2 - (m_2 g) - \lambda \cdot 1 = 0$

I+II: $m_1 \ddot{z}_1 + m_2 \ddot{z}_2 + (m_1+m_2) g - 2\lambda = 0$

$$\Leftrightarrow \quad (m_1-m_2)\ddot{z}_1 + (m_1+m_2) g = 2\lambda$$

I−II: $m_1 \ddot{z}_1 - m_2 \ddot{z}_2 + (m_1 - m_2) g = 0$

$$\Leftrightarrow \quad \ddot{z}_1 = -\frac{m_1-m_2}{m_1+m_2} g$$

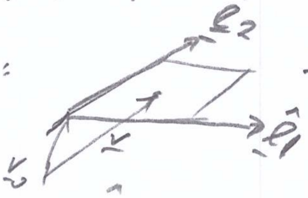
$$\Leftrightarrow \quad \frac{(m_1-m_2)^2}{m_1+m_2} g + (m_1+m_2) g = 2\lambda \Rightarrow \lambda = \frac{g}{2}\,\frac{(m_1+m_2)^2 - (m_1-m_2)^2}{m_1+m_2} = 2g\,\frac{m_1 m_2}{m_1+m_2}$$

(nach oben!)

2.4 Generalisierte Koordinaten

Idee: Finde $f = 3N - 1$ Koordinaten, die unabhängig variierbar sind und dennoch die Zwangsbedingungen erfüllen. Generalisierte Koordinaten $\{q_1, \dots, q_f\}$ $\quad f = \#$ Freiheitsgrade (nur für holonome ZB)

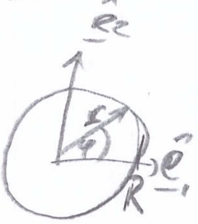
Bsp: (i) Ebene:



$$\underline{r}(t) = \underline{r}_0(t) + q_1 \underline{\hat{e}}_1 + q_2 \underline{\hat{e}}_2 \Rightarrow f = 2$$

mitbewegtes Koordinatensystem

(ii) Kreis:



$$\underline{r}(t) = R(\cos\varphi\, \underline{\hat{e}}_1 + \sin\varphi\, \underline{\hat{e}}_2) \Rightarrow f = 1$$

Ziel: virtuelle Verrückungen $\delta \underline{r}_i$ durch $\delta q_1, \dots \delta q_f$ ausdrücken:

$\underline{r}_i(q_1, \dots, q_f, t)$

$$\delta \underline{r}_i \underset{\delta t = 0}{=} \sum_{j=1}^{f} \left(\frac{\partial}{\partial q_j} \underline{r}_i\right) \delta q_j \Rightarrow \sum_{i=1}^{N} \underline{X}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} \underbrace{\left(\sum_{i=1}^{N} \underline{X}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}\right)}_{} \delta q_j$$

$$= \underbrace{\sum_{j=1}^{f} Q_j \, \delta q_j}_{\text{generalisierte Kräfte}}$$

2.5 Lagrange-Gleichungen 2. Art

D'Alembertsches Prinzip: $\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{i=1}^{N} \underline{X}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} Q_j \, \delta q_j$

linke Seite durch einige Umformungen und Definition der Kinetischen Energie $T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$ zu: $\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right\} \delta q_j$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad \text{mit } j = 1, \dots, f$$

Lagrange-Gleichungen 2. Art

- gilt nur für holonome Zwangsbedingungen (sonst kein generalisierten Koordinaten)
- Für konservative Kräfte: $Q_j = -\frac{\partial V}{\partial q_j}$ mit $V(q_1, \dots, q_f, t)$

$$\Rightarrow \frac{d}{dt} \frac{\partial T - V}{\partial \dot{q}_j} - \frac{\partial T - V}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Lagrange-Funktion