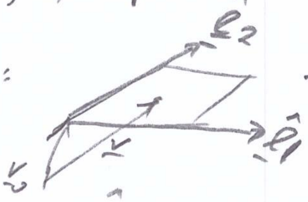
2.4 Generalisierte Koordinaten

Idee: Finde $f = 3N - \Lambda$ Koordinaten, die unabhängig variierbar sind und dennoch die Zwangsbedingungen erfüllen.

Generalisierte Koordinaten $\{q_1, \dots, q_f\}$ $\quad f = \#$ Freiheitsgrade (nur für holonome ZB)

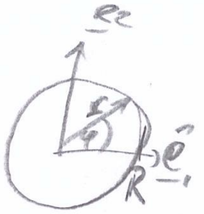
Bsp: (i) Ebene:



$$\underline{r}(t) = \underline{r}_0(t) + q_1 \hat{\underline{e}}_1 + q_2 \hat{\underline{e}}_2 \Rightarrow f = 2$$

mitbewegtes Koordinatensystem

(ii) Kreis:



$$\underline{r}(t) = R\left(\cos\varphi\, \hat{\underline{e}}_1 + \sin\varphi\, \hat{\underline{e}}_2\right) \Rightarrow f = 1$$

Ziel: virtuelle Verrückungen $\delta \underline{r}_i$ durch $\delta q_1, \dots, \delta q_f$ ausdrücken:

$\underline{r}_i(q_1, \dots, q_f, t)$

$$\delta \underline{r}_i \underset{\delta t = 0}{=} \sum_{j=1}^{f} \left(\frac{\partial}{\partial q_j} \underline{r}_i\right) \delta q_j \Rightarrow \sum_{i=1}^{N} \underline{X}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} \underbrace{\left(\sum_{i=1}^{N} \underline{X}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}\right)} \delta q_j$$

$$= \underbrace{\sum_{j=1}^{f} Q_j \, \delta q_j}$$

generalisierte Kräfte

2.5 Lagrange-Gleichungen 2. Art

D'Alembertsches Prinzip: $\displaystyle \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{i=1}^{N} \underline{X}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} Q_j \, \delta q_j$

linke Seite durch einige Umformungen und Definition der kinetischen Energie $\displaystyle T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$ zu: $\displaystyle \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} \left\{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right\} \delta q_j$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad \text{mit } j = 1, \dots, f$$

Lagrange-Gleichungen 2. Art

- gilt nur für holonome Zwangsbedingungen (sonst keine generalisierten Koordinaten)
- Für konservative Kräfte: $\displaystyle Q_j = -\frac{\partial V}{\partial q_j}$ mit $V(q_1, \dots, q_f, t)$

$$\Rightarrow \frac{d}{dt} \frac{\partial T - V}{\partial \dot{q}_j} - \frac{\partial T - V}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Lagrange-Funktion

2.4 Generalized coordinates

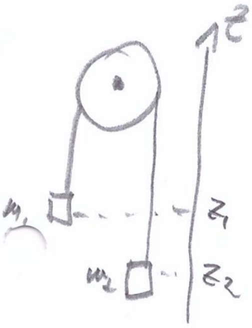
set $\{q_1,\dots,q_f\}$ of independent coordinates that obey given constraints

idea: express virtual displacements $\delta \underline{r}_i$ $(i=1,\dots,N)$ by δq_k $(k=1,\dots,f)$

δx_j $(j=1,\dots,3N)$

with $f = |$degrees of freedom$|$.

case study Atwood's machine



(i) Newton $\underline{F}=m\underline{a}$

(ii) D'Alembert $\sum_{i=1}^{N} \underline{Z}_i \cdot \delta\underline{r}_i = 0 = \sum_{i=1}^{N} (m_i\ddot{\underline{r}}_i - \underline{X}_i)\delta\underline{r}_i$

(iii) Lagrange I $m_j\ddot{x}_j - K_j - \sum_{r=1}^{f} \lambda_r \frac{\partial f_r}{\partial x_j}=0$

$$\ddot{z}_1 = -\frac{m_1-m_2}{m_1+m_2}\, g$$

(iv) Lagrange II: $\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = 0$, $j = 1, \dots, f$

hier $f = 1$ $\Rightarrow$, weil $z_1 + z_2 - l = \text{const}$ $\Rightarrow$ $z_1 = q$, $z_2 = l - q$

$T = \frac{1}{2} \; m_1 \dot z_1^2 + m_2 \dot z_2^2 = \frac{1}{2} (m_1 + m_2) \dot q^2$

$V = m_1 g z_1 + m_2 g z_2 = g\left(m_1 q + m_2 (l - q)\right) = g (m_1 - m_2) q + m g l$

$\Rightarrow L = \frac{1}{2}(m_1 + m_2)\dot q^2 - g (m_1 - m_2) q - m g l$

$\Rightarrow \frac{d}{dt}\left[(m_1 + m_2)\dot q\right] + g (m_1 - m_2) = 0$

$\Leftrightarrow (m_1 + m_2)\ddot q + g (m_1 - m_2) = 0$

$\Leftrightarrow \quad \ddot q = - \frac{m_1 - m_2}{m_1 + m_2} g$

(iv) Lagrange II: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$, $j = 1, \dots, f$

hier $f = 1$, weil $z_1 + z_2 - l = \text{const}$ $\Rightarrow$ $z_1 = q$, $z_2 = l - q$

$T = \frac{1}{2} m_1 \dot{z}_1^2 + m_2 \dot{z}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{q}^2$

$V = m_1 g z_1 + m_2 g z_2 = g\left(m_1 q + m_2(l - q)\right) = g(m_1 - m_2)q + m g l$

$\Rightarrow L = \frac{1}{2}(m_1 + m_2)\dot{q}^2 - g(m_1 - m_2)q - mgl$

$\Rightarrow \frac{d}{dt}\left[(m_1 + m_2)\dot{q}\right] + g(m_1 - m_2) = 0$

$\Leftrightarrow (m_1 + m_2)\ddot{q} + g(m_1 - m_2) = 0$

$\Leftrightarrow \ddot{q} = -\frac{m_1 - m_2}{m_1 + m_2} g$

Einschub: Berechnung der linken Seite

$$\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \left(\frac{\partial}{\partial q_j} \underline{r}_i\right) \delta q_j$$

$$= \sum_{j=1}^{f} \sum_{i=1}^{N} \left\{ \frac{d}{dt}\, m \dot{\underline{r}}_i \left(\frac{\partial}{\partial q_j} \underline{r}_i\right) - m \dot{\underline{r}}_i \underbrace{\frac{d}{dt}\left(\frac{\partial}{\partial q_j} \underline{r}_i\right)} \right\} \delta q_j$$

Mit $\frac{d}{dt}\underline{r}_i \equiv \dot{\underline{r}}_i = \sum_{j=1}^{f} \left(\frac{\partial \underline{r}_i}{\partial q_j}\right) \dot{q}_j + \frac{\partial}{\partial t} \underline{r}_i = \underline{v}_i$ folgt $\frac{d}{dt}\left(\frac{\partial}{\partial q_j} \underline{r}_i\right) = \frac{\partial}{\partial q_j} \underline{v}_i$, und $\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$

denn: $\frac{d}{dt}\left(\frac{\partial}{\partial q_j} \underline{r}_i\right) = \sum_k \left(\frac{\partial^2}{\partial q_k \partial q_j} \underline{r}_i\right) \dot{q}_k + \frac{\partial^2}{\partial t \partial q_j} r_i$

$$\frac{\partial}{\partial q_j} \underline{v}_i = \frac{\partial}{\partial q_j} \left\{ \sum_k \left(\frac{\partial}{\partial q_k} \underline{r}_i\right) \dot{q}_k + \frac{\partial}{\partial t} \underline{r}_i \right\} = \sum_k \left(\frac{\partial^2}{\partial q_j \partial q_k} \underline{r}_i\right) \dot{q}_k + \frac{\partial^2}{\partial q_j \partial t} \underline{r}_i$$

$$\Rightarrow \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_j \sum_i \left\{ \frac{d}{dt} \left(m \underline{v}_i \underbrace{\frac{\partial}{\partial q_j} \underline{r}_i}_{= \frac{\partial}{\partial \dot{q}_j} \underline{v}_i} \right) - m_i \underline{v}_i \left(\frac{\partial}{\partial q_j} \underline{v}_i\right) \right\} \delta q_j \qquad (*)$$

mit $m_i \underline{v}_i \left(\frac{\partial}{\partial \dot{q}_j} \underline{v}_i\right) = \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} m_i v_i^2\right)$ und $m \underline{v}_i \left(\frac{\partial}{\partial q_j} \underline{v}_i\right) = \frac{\partial}{\partial q_j} \left(\frac{1}{2} m_i \underline{v}_i^2\right)$ folgt:

$$(*) = \sum_{j=1} \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right\} \underset{\substack{\uparrow \\ \text{frei variierbar}}}{\delta q_j} = 0, \qquad T \equiv \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$

Für Konservative Kräfte: $Q_j = -\frac{\partial V}{\partial q_j}$

$$\left(V\left(\underline{r}_1(q_1,\dots,q_f,t), \underline{r}_2(q_1,\dots,q_f,t),\dots,\underline{r}_N(q_1,\dots,q_f,t)\right) = V(q_1,\dots,q_f,t)\right)$$

$\Rightarrow \frac{\partial}{\partial \dot{q}_j} V = 0$ im allgemeinen (s.o.)

und der Definition der Lagrange-Funktion $L(\{q_j\}_{j=1}^f, \{\dot{q}_j\}_{j=1}^f, t) = T - V$ folgt:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0} \qquad j = 1, \dots, f$$

Anwendungsschema/Kochrezept:

Kochrezept der Lagrange-Gleichungen 2. Art:

1. Auswählen generalisierter Koordinaten: $q_1, \dots, q_f$, die die (holonomen) Zwangsbedingungen erfüllen: $\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_f, t) \equiv \mathbf{r}_i(q_k, t)$
2. Berechnen von Geschwindigkeiten $\mathbf{v}_i(q_k, \dot{q}_k, t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}_i(q_k, t)$ und kinetischer Energie $T(q_k, \dot{q}_k, t) = \frac{1}{2}\sum_{i=1}^N m_i \mathbf{v}_i(q_k, \dot{q}_k, t)^2$
3. Potenzielle Energie für konservative Kräfte: $V(q_k, t) = V(\mathbf{r}_i(q_k, t))$
$\Rightarrow$Lagrange-Funktion $L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k, t) - V(q_k, t)$
(Potenzielle/Lage-) Energie für nichtkonservative Kräfte: $Q_j = \sum_{i=1}^N \mathbf{X}_i \frac{\partial}{\partial q_j}\mathbf{r}_i(q_1, \dots, q_f, t)$
4. Aufstellung der Bewegungsgleichungen ($j = 1, \dots, f$):
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_j}L\right) - \frac{\partial}{\partial q_j}L = 0 \qquad \text{bzw.} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_j}T\right) - \frac{\partial}{\partial q_j}T = Q_j$$
5. Lösen der Bewegungsgleichungen

Bem: Bei Geschwindigkeitsabhängigen Potenzial – etwa elektromagnetisches Potenzial: $V_{em} = e(\phi - \underline{v}\cdot\underline{A})$ – ergibt sich: $Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}$

Idee: $\sum_{i=1}^N \underline{X}_i\cdot\delta\underline{r}_i = \sum_{i=1}^N \left(-\frac{\partial V}{\partial \underline{r}_i} + \frac{d}{dt}\frac{\partial V}{\partial \dot{\underline{r}}_i}\right)\delta\underline{r}_i$

Für V_{em}: $Q_j = -\frac{\partial V_{em}}{\partial x_j} + \frac{d}{dt}\frac{\partial V_{em}}{\partial \dot{x}_j} = e\left(-\frac{\partial \phi}{\partial x_j} + \dot{\underline{r}}\cdot\frac{\partial \underline{A}}{\partial x_j} - \frac{dA_j}{dt}\right)$

$$= e\left(-\frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t} - \sum_{k=1}^3 \frac{\partial A_j}{\partial x_k}\dot{x}_k + \sum_{k=1}^3 \dot{x}_k\frac{\partial A_k}{\partial x_j}\right)$$

$$= e\left(-\frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t} + \sum_{k=1}^3\left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\right)\dot{x}_k\right) \Leftrightarrow \underline{F}_{em} = e\left(-\nabla\phi - \frac{\partial \underline{A}}{\partial t} + \underline{v}\times(\nabla\times\underline{A}\right.$$

Allgemein: $\sum_{i=1}^{N} \underline{X}_i \cdot \delta \underline{r}_i = \sum_{j=1}^{f} \left(\sum_{i=1}^{N} \underline{X}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j$

$$= \sum_{j=1}^{f} \left(\sum_{i=1}^{N} \left(-\frac{\partial V}{\partial \underline{r}_i} + \frac{d}{dt} \frac{\partial V}{\partial \dot{\underline{r}}_i} \right) \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{j=1}^{f} \left(-\frac{\partial V}{\partial q_j} + \sum_{i=1}^{N} \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{\underline{r}}_i} \right) \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j$$

$$\frac{\partial V}{\partial \dot{q}_j} = \sum_{i=1}^{N} \frac{\partial V}{\partial \dot{\underline{r}}_i} \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j} = \sum_{i=1}^{N} \frac{\partial V}{\partial \dot{\underline{r}}_i} \frac{\partial \underline{r}_i}{\partial q_j}$$

$$= \sum_{j=1}^{f} \underbrace{\left(-\frac{\partial V}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right) \right)} \delta q_j$$

$$= \sum_{j=1}^{f} Q_j \, \delta q_j$$

Zusammenhang: Lagrange I & Energie-Erhaltung:

$$(i)\quad \sum_{i=1}^{N} \underline{F}_i \cdot \dot{\underline{r}}_i = \sum_{i=1}^{N} m \ddot{\underline{r}}_i \cdot \dot{\underline{r}}_i = \frac{d}{dt}\left(\sum_{i=1}^{N} \frac{m_i}{2} \dot{\underline{x}}_i^2\right) = \frac{d}{dt} T$$

$$(ii)\quad \sum_{i=1}^{N} \underline{F}_i \cdot \dot{\underline{r}}_i \underset{\uparrow}{=} -\sum_{i=1}^{N}\left(\nabla_{\underline{r}_i} \sum_{j=1}^{N} V_{ij}(\underset{\uparrow}{r_{ij}}) + \nabla_{\underline{r}_i} V_i(\underline{r}_i)\right) \cdot \dot{\underline{r}}_i$$

$|r_i - r_j|$

Konservative Kräfte

$$= -\frac{d}{dt}\left(\frac{1}{2}\sum_{i,j=1}^{N} V_{ij}(r_{ij}) + \sum_{i=1}^{N} V_i(\underline{r}_i)\right)$$

$$= -\frac{d}{dt} V \qquad \left(\frac{d}{dt} V_{ij}(r_{ij}) = \left(\nabla_{\underline{r}_i} V_{ij}(r_{ij})\right) \cdot \dot{\underline{r}}_i + \left(\nabla_{\underline{r}_j} V_{ij}(r_{ij})\right) \cdot \dot{\underline{r}}_j\right)$$

$$\overset{i+ii}{\Rightarrow} \frac{dT}{dt} + \frac{d}{dt} V = 0$$

(evtl. können dissipative Prozesse, z.B. Reibung, die Energieerhaltung stören/aufheben) $\frac{d}{dt}(T+V) = \sum_{i=1}^{N} \underline{F}_d \cdot \dot{\underline{r}}_i$

Unter Berücksichtigung von Zwangsbedingungen: $\sum_{i=1}^{N} \underline{Z}_i \cdot \dot{\underline{r}}_i$ auf rechter Seite

$$\frac{d}{dt}(T+V) = -\underset{\uparrow}{\sum_{\mu=1}^{\Lambda}} \lambda_\mu \frac{\partial f_\mu}{\partial t}$$

$$\text{wegen } \sum_{i=1}^{N} \underline{Z}_i \cdot \dot{\underline{r}}_i = \sum_{\mu=1}^{\Lambda} \sum_{i=1}^{N} \lambda_\mu \left(\nabla_{\underline{r}_i} f_\mu\right) \cdot \dot{\underline{r}}_i$$

$$\text{und } 0 = \frac{df_\mu}{dt} = \sum_{i=1}^{N} \nabla_{\underline{r}_i} f_\mu + \frac{\partial f_\mu}{\partial t}$$

$\Rightarrow$ Energie ist erhalten, wenn Zwangsbedingungen skleronom sind. $\left(\frac{\partial f_\mu}{\partial t} = 0\right)$

Lagrange II & Newton: (Konservative Kräfte) (1 Punktmasse)

$$T = \frac{1}{2} m \dot{\underline{r}}^2 = \frac{1}{2} m \sum_{j=1}^{3} \dot{x}_i^2, \quad q_1 = x_1,\ q_2 = x_2,\ q_3 = x_3$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left[\left(\frac{1}{2} m \sum_{k=1}^{3} \dot{q}_k^2\right) - V(q_1, q_2, q_3)\right]$$

$$- \frac{\partial}{\partial q_j}\left[T(\dot{q}_1, \dot{q}_2, \dot{q}_3) - V(q_1, q_2, q_3)\right]$$

$$= \frac{d}{dt} m\dot{q}_j + \frac{\partial V}{\partial q_j} = 0$$

$$\Leftrightarrow \quad m\ddot{q}_j = -\frac{\partial V}{\partial q_j} \quad j = 1, 2, 3$$