2.5 Lagrangian equation of 2nd kind

idea: express $\underline{r}_i$, $\dot{\underline{r}}_i$, $\delta\underline{r}_i$ by generalized coordinates q_j, $\dot{q}_j$, δq_j, $j=1,\dots,f$
$i=1,\dots,N$

e.g.: $\delta\underline{r}_i = \sum_{j=1}^{f} \left(\frac{\partial}{\partial q_j}\underline{r}_i\right)\delta q_j$, $\quad \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j} = \frac{\partial \underline{r}_i}{\partial q_j}$, $\quad \frac{d}{dt}\left(\frac{\partial}{\partial q_j}\underline{r}_i\right) = \frac{\partial \dot{\underline{r}}_i}{\partial q_j}$

Defining the Lagrangian / Lagrange function: $L = T - V$
with $T = \sum_{i=1}^{N}\frac{1}{2}m_i\dot{\underline{r}}_i^2$ and $Q_j = -\frac{\partial V}{\partial q_j}$ (generalized forces):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad , j=1,\dots,f$$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

Zusammenhang: Lagrange I & Energie-Erhaltung:

$$(i)\quad \sum_{i=1}^{N} \underline{F}_i \cdot \dot{\underline{r}}_i = \sum_{i=1}^{N} m \ddot{\underline{r}}_i \cdot \dot{\underline{r}}_i = \frac{d}{dt}\left(\sum_{i=1}^{N} \frac{m_i}{2} \dot{\underline{x}}_i^{\,2}\right) = \frac{d}{dt} T$$

$$(ii)\quad \sum_{i=1}^{N} \underline{F}_i \cdot \dot{\underline{r}}_i \underset{\uparrow}{=} -\sum_{i=1}^{N}\left(\underline{\nabla}_{r_i} \sum_{j=1}^{N} V_{ij}(r_{ij}) + \underline{\nabla}_{r_i} V_i(\underline{r}_i)\right)\cdot \dot{\underline{r}}_i$$

$|r_i - r_j|$

Konservative Kräfte

$$= -\frac{d}{dt}\left(\frac{1}{2}\sum_{i,j=1}^{N} V_{ij}(r_{ij}) + \sum_{i=1}^{N} V_i(\underline{r}_i)\right)$$

$$= -\frac{d}{dt} V \qquad \left(\frac{d}{dt} V_{ij}(r_{ij}) = \left(\underline{\nabla}_{r_i} V_{ij}(r_{ij})\right)\cdot \dot{\underline{r}}_i + \left(\underline{\nabla}_{r_j} V_{ij}(r_{ij})\right)\cdot \dot{\underline{r}}_j\right)$$

$$\overset{i+ii}{\Rightarrow} \frac{dT}{dt} + \frac{d}{dt} V = 0$$

(evtl. können dissipative Prozesse, z.B. Reibung, die Energieerhaltung stören/aufheben)

$$\frac{d}{dt}(T+V) = \sum_{i=1}^{N} \underline{F}_d \cdot \dot{\underline{r}}_i$$

Unter Berücksichtigung von Zwangsbedingungen: $\sum_{i=1}^{N} \underline{Z}_i \cdot \dot{\underline{r}}_i$ auf rechter Seite

$$\frac{d}{dt}(T+V) = -\sum_{\mu=1}^{\Lambda} \lambda_\mu \frac{\partial f_\mu}{\partial t}$$

wegen $\sum_{i=1}^{N} \underline{Z}_i \dot{\underline{r}}_i = \sum_{\mu=1}^{\Lambda} \sum_{i=1}^{N} \lambda_\mu \left(\underline{\nabla}_{r_i} f_\mu\right) \cdot \dot{\underline{r}}_i$

und $0 = \frac{df_\mu}{dt} = \sum_{i=1}^{N} \underline{\nabla}_{r_i} f_\mu + \frac{\partial f_\mu}{\partial t} \Leftrightarrow \frac{\partial f_\mu}{\partial t} = -\sum_{i=1}^{N} \left(\underline{\nabla}_{r_i} f_\mu\right)\cdot \dot{\underline{r}}_i$

$\Rightarrow$ Energie ist erhalten, wenn Zwangsbedingungen skleronom sind. $\left(\frac{\partial f_\mu}{\partial t} = 0\right)$

Lagrange II & Newton: (Konservative Kräfte) (1 Punktmasse)

$$T = \frac{1}{2} m \underline{r}^2 = \frac{1}{2} m \sum_{j=1}^{3} \dot{x}_i^{\,2}, \quad q_1 = x_1,\ q_2 = x_2,\ q_3 = x_3$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left[\left(\frac{1}{2} m \sum_{k=1}^{3} \dot{q}_k^{\,2}\right) - V(q_1, q_2, q_3)\right]$$

$$- \frac{\partial}{\partial q_j}\left[T(\dot{q}_1, \dot{q}_2, \dot{q}_3) - V(q_1, q_2, q_3)\right]$$

$$= \frac{d}{dt} m\dot{q}_j + \frac{\partial V}{\partial q_j} = 0$$

$$\Leftrightarrow \quad m\ddot{q}_j = -\frac{\partial V}{\partial q_j} \qquad j = 1, 2, 3$$

$$T(q_k,\dot q_k,t)=\sum_{i=1}^{N}\frac{1}{2}m_i\,\dot{\underline{r}}_i(q_k,\dot q_k,t)^2$$

$$=\sum_{i=1}^{N}\frac{1}{2}m_i\left[\sum_{j=1}^{f}\frac{\partial \underline{r}_i}{\partial q_j}\dot q_j+\frac{\partial \underline{r}_i}{\partial t}\right]^2$$

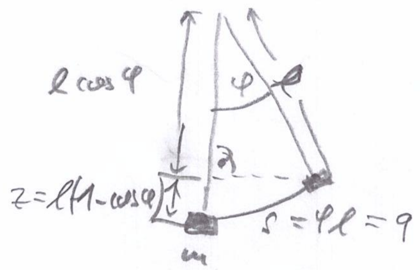
$$=a+\sum_{j=1}^{f}b_j\dot q_j+\sum_{j,m=1}^{f}c_{jm}\dot q_j\dot q_m$$

Sortieren nach Ordnung in $\dot q_j$

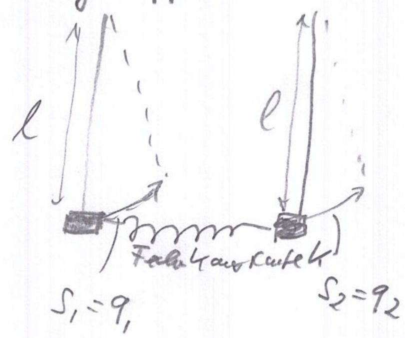
Nur für skleronome Zwangsbedingungen ist T eine homogene Bilinearform ($a=b_j=0$) in $\dot q_j$ ($j=1,\dots,f$)

2.6 Normalschwingungen

a) 1 Pendel



b) 2 gekoppelte Pendel



Neue Sichtweise:
- holonome, skleronome Zwangsbedingungen

Annahme: Potenzial V mit stabiler Ruhelage
- Wahl der generalisierten Koordinaten $q_1,\dots,q_f$ mit Ruhelage $q_1=\dots=q_f=0$
- Entwicklung der potenziellen Energie um die Ruhelage:

$$V(q_1,\dots,q_f)=\underbrace{V(0,\dots,0)}_{\substack{\text{o.B.d.A.}=0\\(\text{Offset der Energie})}}+\sum_{j=1}^{f}\underbrace{\left(\frac{\partial V}{\partial q_j}\right)_0}_{=-Q_j\overset{!}{=}0}q_j+\frac{1}{2}\sum_{j,k=1}^{f}\underbrace{\left(\frac{\partial^2 V}{\partial q_j\partial q_k}\right)_0}_{=V_{jk}}q_jq_k+\dots$$

Ruhelage

$$\Rightarrow V(q_1,\dots,q_f)\approx\frac{1}{2}\sum_{j,k=1}^{f}V_{jk}q_jq_k\geq 0\qquad\text{mit } V_{jk}=V_{kj}$$

pos. definite quadratische Form, da Ruhelage stabil & $T\geq 0$

- kinetische Energie: $T=\frac{1}{2}\sum_{j,k=1}^{f}T_{jk}\dot q_j\dot q_k\geq 0$ mit $T_{jk}=T_{kj}=\sum_{i=1}^{N}m_i\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)_0\left(\frac{\partial \underline{r}_i}{\partial q_k}\right)_0$

Fall a) $T = \frac{1}{2} m \dot{q}^2$

$$V(q) = mg\, z(q) = mg\ell\left(1-\cos\varphi(q)\right) = mg\ell\left(1-\cos\frac{q}{\ell}\right)$$

$$\left.\begin{aligned} \Rightarrow \left(\frac{\partial V}{\partial q}\right)_0 &= mg\frac{\ell}{\ell}\sin\frac{q}{\ell}\Big|_0 = 0 \\ \left(\frac{\partial^2 V}{\partial q^2}\right)_0 &= m\frac{g}{\ell}\cos\frac{q}{\ell}\Big|_0 = m\frac{g}{\ell} \end{aligned}\right\} \Rightarrow V(q) \approx \frac{1}{2}\left(\frac{\partial^2 V}{\partial q^2}\right)_0 q^2 = \frac{1}{2} m \frac{g}{\ell} q^2$$

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} \overset{L=T-V}{=} \frac{d}{dt}\left(m\dot{q}\right) + m\frac{g}{\ell}q$$

$$= m\ddot{q} + m\frac{g}{\ell}q$$

$$\Leftrightarrow \quad 0 = \ddot{q} + \frac{g}{\ell}q \quad \text{oder} \quad 0 = \ddot{q} + \omega^2 q \quad \text{mit } \omega^2 = \frac{g}{\ell}$$

Fall b): $T = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2\right)$

$$V(q_1, q_2) = mg\ell\left(1-\cos\frac{q_1}{\ell}\right) + mg\ell\left(1-\cos\frac{q_2}{\ell}\right) + \underbrace{\frac{k}{2}(q_1-q_2)^2}_{\text{Hooke'sches Gesetz}}$$

$$\left(\frac{\partial^2 V}{\partial q_1^2}\right)_0 = \left(\frac{\partial^2 V}{\partial q_2^2}\right)_0 = m\frac{g}{\ell} + k$$

$$\left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 = \underbrace{mg\frac{\partial}{\partial q_1}\left(\sin\frac{q_2}{\ell}\right)}_{=0} - k^2\frac{\partial}{\partial q_1}(q_1 - q_2) = -k = \left(\frac{\partial^2 V}{\partial q_2 \partial q_1}\right)_0$$

$$\Rightarrow V(q_1, q_2) \approx \frac{1}{2} m \frac{g}{\ell}\left(q_1^2 + q_2^2\right) + \frac{1}{2} k (q_1 - q_2)^2$$

In Matrixnotation:

$$\underline{\underline{T}} \equiv \left\{T_{jk}\right\}_{j,k=1,2} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \left\{V_{jk}\right\}_{j,k=1,2} = \begin{pmatrix} m\frac{g}{\ell}+k & -k \\ -k & m\frac{g}{\ell}+k \end{pmatrix} \equiv \underline{\underline{V}}$$

Bewegungsgleichungen: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1,2$

$$\left.\begin{aligned} m\ddot{q}_1 + m\frac{g}{\ell}q_1 + k(q_1 - q_2) &= 0 \\ m\ddot{q}_2 + m\frac{g}{\ell}q_2 - k(q_1 - q_2) &= 0 \end{aligned}\right\} \text{lineare gekoppelte DGLs}$$