Energy conservation:

$$\underbrace{\frac{d}{dt}(T+V) = -\sum_{\mu=1}^{\Lambda} \lambda_\mu \frac{\partial f_\mu}{\partial t}}$$

$= 0$ for scleronomic constraints

↳ $f_\mu(r_1, \ldots, r_N, \cancel{t})$ not explicitly time-dependent

2.6 Normal modes

a) 1 pendulum

b) 2 coupled pendula

idea: expand potential V at points of rest (set to $q_1 = q_2 = \ldots = q_f = 0$)

↳ scaling of energy: $V(0, \ldots, 0) = 0$

$$\hookrightarrow V(q_1, \ldots, q_f) \approx \frac{1}{2} \sum_{j,k=1}^{f} \underbrace{V_{jk}}_{\left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_0} q_j q_k \geq 0$$

- Kinetic energy:

$$T(q_k, \dot{q}_k, t) = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i (q_k, \dot{q}_k, t)^2$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i \left[\sum_{j=1}^{f} \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right]^2$$

$$= a + \sum_{j=1}^{f} b_j \dot{q}_j + \sum_{j,m=1}^{f} c_{jm} \dot{q}_j \dot{q}_m$$

for scleronomic constraints $a = b_j = 0$)

$$\Rightarrow T(\dot{q}_1, \ldots, \dot{q}_f) = \frac{1}{2} \sum_{j,k=1}^{f} \underbrace{T_{jk}}_{\approx \sum_{i=1}^{N} m_i \left(\frac{\partial r_i}{\partial q_j}\right)_0 \left(\frac{\partial r_i}{\partial q_k}\right)_0} \dot{q}_j \dot{q}_k \geq 0$$

a) $T = \frac{1}{2} m \dot{q}^2$, $V = \frac{1}{2} m \frac{g}{\ell} q^2$

$$\Rightarrow 0 = \ddot{q} + \frac{g}{\ell} q$$

b) $T = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2\right)$

$$V = \frac{1}{2} m \frac{g}{l}\left(q_1^2 + q_2^2\right) + \frac{1}{2} k \left(q_1 - q_2\right)^2$$

matrix form: $\underline{\underline{T}} = \{T_{jk}\}_{j,k=1,2} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

$$\underline{\underline{V}} = \{V_{jk}\}_{j,k=1,2} = \begin{pmatrix} m\frac{g}{l} + k & -k \\ -k & m\frac{g}{l} + k \end{pmatrix}$$

equation of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad , j = 1,2$

$$m\ddot{q}_1 + m\frac{g}{l} q_1 + k(q_1 - q_2) = 0$$

$$m\ddot{q}_2 + m\frac{g}{l} q_2 - k(q_1 - q_2) = 0$$

using the ansatz: $q_j = A_j e^{i\omega t}$:

eigenvalue equation: $\underbrace{\begin{pmatrix} m\frac{g}{l} + k & -k \\ -k & m\frac{g}{l} + k \end{pmatrix}}_{\underline{\underline{V}}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \underbrace{m\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}_{\omega^2 \underline{\underline{T}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}$

Lösungsansatz: $q_j = A_j e^{i\omega t} \Rightarrow \ddot{q}_j = -\omega^2 A_j e^{i\omega t}$

$\Rightarrow$ Eigenwertgleichung:

$$\underbrace{\begin{pmatrix} m\frac{g}{l}+k-m\omega^2 & -k \\ -k & m\frac{g}{l}+k-m\omega^2 \end{pmatrix}}_{=\underline{\underline{V}}-\omega^2 \underline{\underline{T}}}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$\Rightarrow$ charakteristische Gleichung:

$$0 = m^2 \begin{vmatrix} \frac{g}{l}+\frac{k}{m}-\omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l}+\frac{k}{m}-\omega^2 \end{vmatrix}$$

$$= \omega^4 - 2\omega^2\left(\frac{g}{l}+\frac{k}{m}\right) + \left(\frac{g}{l}+\frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2$$

$$\Rightarrow \omega_{1,2}^2 = \left(\frac{g}{l}+\frac{k}{m}\right) \pm \frac{k}{m} = \begin{cases} \frac{g}{l} \\ \frac{g}{l}+2\frac{k}{m} \end{cases}$$

Eigenfrequenzen: $\omega_1 = \sqrt{\frac{g}{l}}$: ungestörte Eigenfrequenz

$$\omega_2 = \sqrt{\frac{g}{l}+\frac{2k}{m}} =: \sqrt{\omega_0^2 + 2\tilde{\omega}^2}$$

Eigenvektoren:

zu ω_1: $kA_1^{(1)} - kA_2^{(1)} = 0 \Rightarrow \begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \underset{\uparrow}{=} \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Normierung $m\sum_k |A_k|^2 = 1 = \sum_{jk} A_j T_{jk} A_k$

zu ω_2: $-kA_1^{(1)} - kA_2^{(2)} = 0 \Rightarrow \begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} \overset{\downarrow}{=} \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

allgemeine Lsg:

$$\begin{pmatrix} A_1^{(1)} & A_2^{(1)} \\ A_1^{(2)} & A_2^{(2)} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{2m}}\begin{pmatrix} q_1+q_2 \\ q_1-q_2 \end{pmatrix}$$

← Schwerpunktkoordinate

← Relativkoordinate

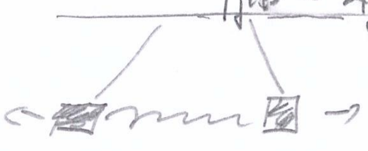
$= \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \Rightarrow$ Umkehrung auch möglich

$q_j(t) \sim A_j^{(1)}Q_1 + A_j^{(2)}Q_2$

Normalschwingungen:

$\omega_1 = \sqrt{\frac{g}{l}}$

gleichphasig



$\omega_2 = \sqrt{\frac{g}{l}+2\frac{k}{m}}$

gegenphasig

$\Rightarrow$ Überlagern: Schwebung

Normalschwingungen ganz allgemein:

$$L = T - V = \frac{1}{2}\sum_{j,k=1}^{f}\left(T_{jk}\,\dot q_j \dot q_k - V_{jk}\, q_j q_k\right)$$

$$\hookrightarrow \frac{\partial L}{\partial \dot q_\ell} = \frac{1}{2}\sum_{j,k=1}^{f} T_{jk}\underbrace{\frac{\partial}{\partial \dot q_\ell}(\dot q_j \dot q_k)} = \frac{1}{2}\sum_k T_{\ell k}\dot q_k + \frac{1}{2}\underbrace{\sum_j T_{j\ell}\dot q_j}_{=\sum_k T_{\ell k}\dot q_k} = \sum_k T_{\ell k}\dot q_k$$

$$= \frac{\partial \dot q_j}{\partial \dot q_\ell}\dot q_k + \dot q_j\frac{\partial \dot q_k}{\partial \dot q_\ell}$$

$$= \delta_{j\ell}\,\dot q_k + \dot q_j\,\delta_{k\ell}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_\ell}\right) = \sum_k T_{\ell k}\ddot q_k$$

$$\frac{\partial V}{\partial q_\ell} = \frac{1}{2}\sum_{j,k=1}^{f} T_{jk}\frac{\partial}{\partial q_\ell}(q_j q_k) = \ldots = \sum_k V_{\ell k} q_k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\ell}\right) - \frac{\partial L}{\partial q_\ell} = 0 \Leftrightarrow \sum_k\left(T_{\ell k}\ddot q_k + V_{\ell k} q_k\right) = 0\,,\ \ell = 1,\ldots,f$$

lineare DGLs!

Ansatz: $q_k(t) = A_k e^{i\omega t}$

$$\Rightarrow \sum_{k=1}^{f}\left(V_{\ell k} - \omega^2 T_{\ell k}\right)A_k = 0$$

- Eigenwertgleichung für ω^2! $\omega_\alpha^2, \alpha = 1,\ldots,f$
- homogenes, lineares GLS für $A_k \Rightarrow A_k^{(\alpha)}$

Nichttriviale Lsg genau dann, wenn $\det\left(V_{\ell k} - \omega^2 T_{\ell k}\right) = 0$

Charakt. Gleichung für ω^2

allg. Lsg: $q_k(t) = \mathrm{Re}\left\{\sum_{\alpha=1}^{f} C_\alpha A_k^{(\alpha)} e^{i\omega_\alpha t}\right\}$

Bestimmt durch Anfangsbed. $q_k(0), \dot q_k(0)$

V, T pos definit $\Rightarrow \omega^2 > 0$

$$\sum_{k=1}^{f}\left(V_{\ell k} - \omega^2 T_{\ell k}\right)A_k = 0 \quad \Big|\cdot \sum_{\ell=1}^{f} A_\ell^*$$

$$\sum_{\ell k=1}^{f} V_{\ell k} A_\ell^* A_k - \omega^2\sum_{\ell k=1}^{f} T_{\ell k} A_\ell^* A_k = 0 \Rightarrow \omega^2 = \frac{\sum_{\ell k=1}^{f} V_{\ell k} A_\ell^* A_k}{\sum_{\ell k=1}^{f} T_{\ell k} A_\ell^* A_k}$$

Normalkoordinaten:

Idee: Transformiere auf neue Koordinaten Q_α, sodass die Bewegungsgleichungen entkoppeln $\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0, \alpha = 1,\dots,f$

⇒ Hauptachsentrafo der symmetrischen Matrizen $V_{\ell k}$, $T_{\ell k}$

⇒ Diagonalisierung durch Trafo mit (reell gewählten) Eigenvektoren $A_k^{(\alpha)}$:

$$q_k(t) = \sum_{\alpha=1}^{f} A_k^{(\alpha)} Q_\alpha(t) \Leftrightarrow \underline{q} = \underline{\underline{A}}\,\underline{Q}$$

Check Diagonalform von $V_{\ell k}, T_{\ell k}$ (gleichzeitig diagonalisierbar)

$$\mathrm{I}: \sum_{k=1}^{f} (V_{\ell k} - \omega_\alpha^2 T_{\ell k}) A_k^{(\alpha)} = 0 \qquad \Big| \cdot \sum_{\ell=1}^{f} A_\ell^{(\beta)}$$

$$\mathrm{II}: \sum_{\ell=1}^{f} (V_{k\ell} - \omega_\beta^2 T_{k\ell}) A_\ell^{(\beta)} = 0 \qquad \Big| \cdot \sum_{k=1}^{f} A_k^{(\alpha)}$$

$$\Rightarrow \mathrm{I}-\mathrm{II}: \sum_{k\ell=1}^{f} \underbrace{(V_{\ell k} - V_{k\ell})}_{=0} A_\ell^{(\beta)} A_k^{(\alpha)} - (\omega_\alpha^2 T_{\ell k} - \omega_\beta^2 T_{k\ell}) A_\ell^{(\beta)} A_k^{(\alpha)} = 0$$

$$\Rightarrow \underbrace{(\omega_\alpha^2 - \omega_\beta^2)}_{\neq 0 \text{ für } \alpha \neq \beta} \underbrace{\sum_{k\ell=1}^{f} A_\ell^{(\beta)} T_{\ell k} A_k^{(\alpha)}}_{T^{(\beta\alpha)} \overset{!}{=} 0 \text{ für } \alpha\neq\beta} = 0$$

(transformierte kinetische Energie)

$$\Rightarrow \sum_{k\ell=1}^{f} A_\ell^{(\beta)} T_{\ell k} A_k^{(\alpha)} = \delta_{\beta\alpha}$$

(geeignete Normierung der Eigenvektoren A_k für $\alpha=\beta$)

$$\Rightarrow \sum_{k\ell=1}^{f} A_\ell^{(\beta)} V_{\ell k} A_k^{(\alpha)} = \omega_\alpha^2 \sum_{k\ell=1}^{f} A_\ell^{(\beta)} T_{\ell k} A_k^{(\alpha)} = \omega_\alpha^2 \delta_{\beta\alpha}$$

⇒ $T_{\ell k}$ & $V_{\ell k}$ gleichzeitig diagonalisiert

$$\Rightarrow L = \frac{1}{2} \sum_{jk=1}^{f} (T_{jk} \dot{q}_j \dot{q}_k - V_{jk} q_j q_k)$$

$$= \frac{1}{2} \sum_{\alpha\beta=1}^{f} \Big(\underbrace{\sum_{jk=1} A_j^{(\beta)} T_{jk} A_k^{(\alpha)}}_{\delta_{\beta\alpha}} \dot{Q}_\alpha \dot{Q}_\beta - \underbrace{\sum_{jk=1} A_j^{(\beta)} V_{jk} A_k^{(\alpha)}}_{=\omega_\alpha^2 \delta} Q_\alpha Q_\beta \Big)$$

$$= \frac{1}{2} \sum^{f} (\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2) \Rightarrow \text{Entkopplung: } \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Q}_\alpha} - \frac{\partial \mathcal{L}}{\partial Q_\alpha} = 0 = \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$$