4 Symmetries & laws of conservation

4.1 Noether-Theorem

idea: continuous infinitesimal transformation that leave the physical system invariant $\left(\frac{dL}{ds}=0\right) \Longleftrightarrow$ conserved quantity

Theorem: Let the Lagrange function/Lagrangian $L(q_1,\dots,q_f,\dot{q}_1,\dots,\dot{q}_f)$ of an autonomous system be invariant w.r.t. infinitesimal transformations $q \longrightarrow \underline{h}^s(q)$ (s: continuous parameter, $\underline{h}^{s=0}(q)=q$)

Then there exists an integral of motion / constant / conserved quantity:

$$I(q,\dot{q}) = \sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j}\left(\frac{d}{ds} h_j^s(q_j)\right)\Big|_{s=0} .$$

- $\underline{h}^s(q) = \underbrace{\underline{h}^{s=0}(q)}_{=q} + s\left(\frac{d}{ds}\underline{h}^s(q)\right)\Big|_{s=0} + O(s^2)$

4.2 Translational invariance

translational invariance $\longleftrightarrow$ conservation of momentum

$$\left.\begin{array}{l} \text{e.g.: } h^s: \underline{r}_i \rightarrow \underline{r}_i + s\,\underline{e}_x \\ \text{and } \dfrac{dL}{ds}=0 \end{array}\right\} \Rightarrow \sum_i m_i \dot{x}_i = P_x = \text{const.}$$

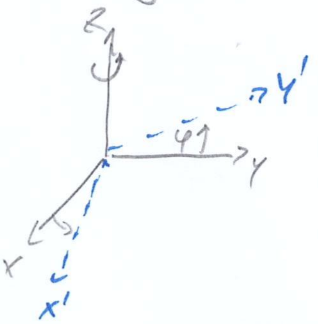
x-component of total momentum.

- generalized momenta: $P_j = \frac{\partial L}{\partial \dot{q}_j}$
- cyclic coordinates: $\frac{\partial L}{\partial q_j} = 0 \overset{\uparrow}{\Rightarrow} P_j = \text{constant}$

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}$$

4.3 Rotationsinvarianz

Drehung um z-Achse (Winkel $\varphi = s$)

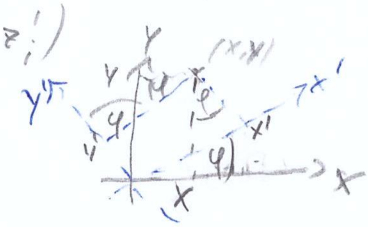


$$\underline{h}^s : \underline{r}_i = (x_i, y_i, z_i) \longrightarrow \underline{r}_i' = (x_i', y_i', z_i')$$

mit (*) $x_i' = x_i \cos s + y_i \sin s$

$y_i' = -x_i \sin s + y_i \cos s$

$z_i' = z_i$



$$\Leftrightarrow \begin{pmatrix} x_i' \\ y_i' \\ z_i' \end{pmatrix} = \begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \underset{s \ll 1}{\approx} \Bigg[\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbb{1}} + \underbrace{\begin{pmatrix} 0 & s & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{(-s J_z)} \Bigg] \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

**

$\boxed{\begin{aligned} &(*)\; \left.\begin{aligned} x &= r\cos\alpha \\ y &= r\sin\alpha \end{aligned}\right\} \Rightarrow x' = r\cos(\alpha - s) = \underbrace{r\cos\alpha}_{x}\cos s + \underbrace{r\sin\alpha}_{y}\sin s \\ &\qquad\qquad\qquad\quad y' = r\sin(\alpha - s) = \underbrace{r\sin\alpha}_{y}\cos s - \underbrace{r\cos\alpha}_{x}\sin s \\ &\qquad\qquad \Rightarrow x' = x\cos s + y\sin s \\ &\qquad\qquad\quad\; y' = -x\sin s + y\cos s \end{aligned}}$

Erzeugende für infinitesimale Drehung um z-Achse $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

** Koordinatensystem drehen

$$= \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} + s \begin{pmatrix} y_i \\ -x_i \\ 0 \end{pmatrix} = \underline{r}_i + s(\underline{r}_i \times \underline{e}_z)$$

$$\underline{r}_i' = \underbrace{\underline{h}^s(\underline{r}_i)\Big|_{s=0}}_{\underline{r}_i} + s\left(\frac{d}{ds}\underline{h}^s(\underline{r}_i)\right)_{s=0} + O(s^2)$$

mit $\frac{d}{ds}\left(\underline{h}^s(\underline{r}_i)\right)_{s=0} = \underline{r}_i \times \underline{e}_x$

$$\Rightarrow L(\underline{r}_i, \dot{\underline{r}}_i) \underset{\text{invariant}}{=} L(\underline{r}_i' = \underline{h}^s(\underline{r}_i), \dot{\underline{r}}_i')$$

T ist rotationsinvariant (hängt nur von $|\dot{\underline{r}}_i|$ ab) (Drehung: orthogonale Trafo)

$$\Rightarrow \left(\frac{dL}{ds}\right)_{s=0} = -\left(\frac{dV}{ds}\right)_{s=0} = -\sum_{i=1}^{N} \underbrace{(\nabla_{\underline{r}_i'} V)}_{-\underline{F}_i} \underbrace{\left(\frac{d\underline{r}_i'}{ds}\right)_{s=0}}_{\left(\frac{d\underline{h}^s}{ds}\right)_{s=0}} = \sum_{i=1}^{N} \underline{F}_i \cdot (\underline{r}_i \times \underline{e}_z)$$

$$\underset{\text{zykl. Permutation}}{=} \underline{e}_z \cdot \sum_{i=1}^{N} \underline{F}_i \times \underline{r}_i = -\underline{e}_z \cdot \underbrace{\sum_{i=1}^{N} (\underline{r}_i \times \underline{F}_i)}_{\text{Gesamtdrehmoment}} \overset{!}{=} 0$$

$$\Rightarrow \text{Integral der Bewegung: } I = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{\underline{r}}_i}\left(\frac{d\underline{h}^s}{ds}\right)_{s=0} = \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot (\underline{r}_i \times \underline{e}_z)$$

$$= \underline{e}_z \cdot \sum_{i=1}^{N} (m_i \dot{\underline{r}}_i \times \underline{r}_i) = -\underline{e}_z \cdot \underbrace{\sum_{i=1}^{N} (\underline{r}_i \times m_i \dot{\underline{r}}_i)}_{\text{Drehimpuls } \underline{l}} = -l_z$$

Rotationsinvarianz $\Rightarrow$ Drehimpulserhaltung

4.4 Zeitliche Translationsinvarianz

Zeitliche Translationsinvarianz:

(i) Zwangsbedingungen nicht explizit t-abhängig

$$\Rightarrow \underline{r}_i = \underline{r}_i(q_1,\dots,q_f)\,,\quad \frac{\partial \underline{r}_i}{\partial t} = 0 \Rightarrow \dot{\underline{r}}_i = \sum_{j=1}^{f} \frac{\partial \underline{r}_i}{\partial q_j}\,\dot q_j$$

(ii) $\dfrac{\partial L}{\partial t} = 0$

• Beobachtung: $$\sum_{j=1}^{f}\left(\frac{\partial L}{\partial \dot q_j}\right)\dot q_j \underset{\uparrow}{=} \sum_{j=1}^{f}\left(\frac{\partial T}{\partial \dot q_j}\right)\dot q_j \left(= \underbrace{\sum_{j=1}^{f} m_j\,\dot q_j\,\dot q_j}_{\text{Achtung! siehe 32a}} \right) \underset{\uparrow}{=} 2T \quad (*)$$

$\left(L = T(\dot q) - V(q), \quad T = \frac{1}{2}\sum_{j=1}^{f} m_j \dot q_j^{2} \leftarrow \text{gilt nicht immer!} \right)$ genauer! (Eulerscher Satz)

• $$\frac{d}{dt}L = \sum_{j=1}^{f}\left(\frac{\partial L}{\partial \dot q_j}\,\underbrace{\ddot q_j}_{\frac{d\dot q_j}{dt}} + \underbrace{\frac{\partial L}{\partial q_j}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)}\dot q_j \right) + \underbrace{\frac{\partial L}{\partial t}}_{\overset{(ii)}{=}0}$$

$$= \sum_{j=1}^{f}\left(\frac{\partial L}{\partial \dot q_j}\,\ddot q_j + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)\dot q_j \right)$$

$$= \frac{d}{dt}\left(\sum_{j=1}^{f} \frac{\partial L}{\partial \dot q_j}\,\dot q_j \right) \overset{(*)}{=} 2\,\frac{dT}{dt}$$

$$\Rightarrow \quad 0 = \frac{d}{dt}(2T - L) \underset{L = T - V}{=} \frac{d}{dt}(T + V) \Rightarrow T + V = \text{constant}$$

Zeittranslationsinvarianz $\Rightarrow$ Energieerhaltung

skleronome Zwangsbedingungen, $\dfrac{\partial L}{\partial t} = 0 \Rightarrow E = T + V = \text{const.}$

Kinetische Energie: Nachtrag

$$T = \frac{1}{2}\sum_i m_i \dot{\underline{r}}_i^{\,2} = \frac{1}{2}\sum_{jk} T_{jk}\, \dot{q}_j \dot{q}_k$$

mit $T_{jk} = \sum_{i=1}^{N} m_i \left(\frac{\partial \underline{r}_i}{\partial q_j}\right)\left(\frac{\partial \underline{r}_i}{\partial q_k}\right)$ abhängig von $q_1, \dots, q_f$!

T ist homogen quadratische Funktion in $\dot{q}_1, \dots, \dot{q}_f$:

$$T(\lambda \dot{q}_1, \dots, \lambda \dot{q}_f) = \lambda^2 T(\dot{q}_1, \dots, \dot{q}_f)$$

$\frac{\partial}{\partial \lambda}\Big|_{\lambda=1}$

$$\Rightarrow \sum_k \frac{\partial T}{\partial(\lambda \dot{q}_k)} \underbrace{\frac{\partial(\lambda \dot{q}_k)}{\partial \lambda}}_{\dot{q}_k}\Bigg|_{\lambda=1} = 2\lambda T(\dot{q}_1, \dots, \dot{q}_f)\Big|_{\lambda=1}$$

$$\sum_{k=1}^{f} \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

Bsp. $\underline{\underline{T}}$ in Kugelkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \sin^2\vartheta \end{pmatrix}$

$\underline{\underline{T}}$ in Zylinderkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

da V unabhängig von $\dot{q}_1, \dots, \dot{q}_f$ ist gilt: $\sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \dot{q}_k = 2T$

Γ • Betrachte: $\bar{L}(\underbrace{q_j, t}_{q_1,\dots,q_f, q_{f+1}}, \frac{dq_j}{d\tau}, \frac{dt}{d\tau}) := L(q_j, \frac{1}{\frac{dt}{d\tau}} \frac{dq_j}{d\tau}, t) \frac{dt}{d\tau}$

also $t \to$ q-artige Variable: $q_j = q_j(\tau)$, $t = t(\tau)$, $t = q_{f+1}$, $\frac{dt}{d\tau} = \mathring{q}_{f+1}$

$\bar{L}$ sei invariant unter Zeittrafo: $h^s(q,t) = (q, t+s)$

$\Rightarrow$ (i) Hamiltonsches Prinzip für $\bar{L}$

$$0 = \delta \int_{\tau_1}^{\tau_2} d\tau\, \bar{L} = \delta \int_{t_1}^{t_2} dt\, L \Leftrightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

(ii) Noether-Theorem für $\bar{L}$:

Integral der Bewegung: $$I = \sum_{j=1}^{f+1} \frac{\partial \bar{L}}{\partial \mathring{q}_j} \underbrace{\left(\frac{d}{ds} h_i^s(q_1,\dots,q_f, \overset{t}{\downarrow}{q_{f+1}}) \right)\Big|_{s=0}}_{(0,\dots,0,1)} = \frac{\partial \bar{L}}{\partial \mathring{q}_{f+1}}$$

$$\underset{\substack{q_{f+1} = t(\tau) \\ \mathring{q}_{f+1} = \frac{dt}{d\tau}}}{=} \frac{\partial \bar{L}}{\partial (\frac{dt}{d\tau})} = L + \sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} \underbrace{\frac{\partial}{\partial(\frac{dt}{d\tau})} \left(\frac{1}{\frac{dt}{d\tau}} \frac{dq_j}{\partial \tau} \right)}_{= -\frac{1}{(\frac{dt}{d\tau})^2} \frac{dq_j}{\partial \tau}} \frac{dt}{\partial \tau}$$

$$= L - \underbrace{\sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j}_{=2T} = T - V - 2T = -(T+V)$$

$\lrcorner$