

4.1 Noether - Theorem

idea: continuous infinitesimal transformations that leave the physical system invariant ($\frac{dL}{ds} = 0$) $\Leftrightarrow$ conserved quantity

Theorem: Let the Lagrange function/Lagrangian $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$ of an autonomous system be invariant w.r.t. infinitesimal transformations $q \rightarrow h^s(q)$ (s : continuous parameter, $h^{s=0}(q) = q$)
Then there exists an integral of motion / conserved quantity:

$$I(q, \dot{q}) = \sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \left(\frac{d}{ds} h_j^s(q) \right) \Big|_{s=0}$$

$$\bullet h^s(q) = \underbrace{h^{s=0}(q)}_{=q} + s \left(\frac{d}{ds} h^s(q) \right) \Big|_{s=0} + O(s^2)$$

4.2 Translational invariance

translational invariance $\Leftrightarrow$ conservation of momentum

$$\left. \begin{array}{l} \text{e.g.: } h^s = \underline{r}_i \rightarrow \underline{r}_i + s \underline{e}_x \\ \text{and } \frac{dL}{ds} = 0 \end{array} \right\} \Rightarrow \sum_i m_i \dot{x}_i = P_x = \text{const.}$$

x -component of total momentum.

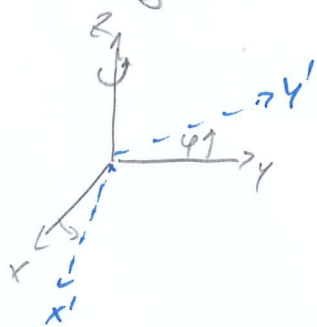
$$\bullet \text{ generalized momenta: } P_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$\bullet \text{ cyclic coordinates: } \frac{\partial L}{\partial \dot{q}_j} = 0 \Rightarrow P_j = \text{constant}$$

$\uparrow$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}_j}$$

Drehung um z-Achse (Winkel $\varphi = S$)

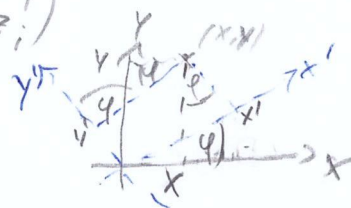


$$h^S: \underline{r}_i = (x_i, y_i, z_i) \rightarrow \underline{r}_i' = (x_i', y_i', z_i')$$

$$\text{mit } x_i' = x_i \cos S + y_i \sin S$$

$$y_i' = -x_i \sin S + y_i \cos S$$

$$z_i' = z_i$$



$$\Rightarrow \begin{pmatrix} x_i' \\ y_i' \\ z_i' \end{pmatrix} = \begin{pmatrix} \cos S & \sin S & 0 \\ -\sin S & \cos S & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & S & 0 \\ -S & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

$$\textcircled{2} \begin{cases} x = r \cos \alpha \\ y = r \sin \alpha \end{cases} \Rightarrow \begin{cases} x' = r \cos(\alpha - S) = r \cos \alpha \cos S + r \sin \alpha \sin S \\ y' = r \sin(\alpha - S) = r \sin \alpha \cos S - r \cos \alpha \sin S \end{cases}$$

Erzeugende für rotationsinvariant

Drehung um z-Achse $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

xx
Koordinaten-
systeme

$$\Rightarrow \begin{cases} x' = x \cos S + y \sin S \\ y' = -x \sin S + y \cos S \end{cases}$$

$$= \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} + S \begin{pmatrix} y_i \\ -x_i \\ 0 \end{pmatrix} = \underline{r}_i + S(\underline{r}_i \times \underline{e}_z)$$

$$\underline{r}_i' = \underline{h}^S(\underline{r}_i) \Big|_{S=0} + S \left(\frac{d}{ds} \underline{h}^S(\underline{r}_i) \right) \Big|_{S=0} + O(S^2)$$

$$\text{mit } \frac{d}{ds} \left(\underline{h}^S(\underline{r}_i) \right) \Big|_{S=0} = \underline{r}_i \times \underline{e}_z$$

T ist rotationsinvariant

(hängt nur von $|\underline{r}_i|$ ab)

(Drehung: orthogonale Transformation)

$$\Rightarrow L(\underline{r}_i, \dot{\underline{r}}_i) = L(\underline{r}_i' = \underline{h}^S(\underline{r}_i), \dot{\underline{r}}_i)$$

$$\Rightarrow \left(\frac{dL}{ds} \right)_{S=0} = - \left(\frac{dV}{ds} \right)_{S=0} = - \sum_{i=1}^N \underbrace{(\underline{r}_i' \cdot \underline{V})}_{-\underline{r}_i \cdot \underline{V}} \underbrace{\left(\frac{d\underline{r}_i'}{ds} \right)_{S=0}}_{\left(\frac{d\underline{h}^S}{ds} \right)_{S=0}} = \sum_{i=1}^N \underline{r}_i \cdot \left(\underline{r}_i \times \underline{e}_z \right)$$

$$= \underline{e}_z \cdot \sum_{i=1}^N \underline{r}_i \times \underline{r}_i = - \underline{e}_z \cdot \sum_{i=1}^N (\underline{r}_i \times \underline{r}_i) = 0$$

zykl. Permutation

Gr. auf Drehmoment

$$\Rightarrow \text{Integral der Bewegung: } I = \sum_{i=1}^N \frac{\partial L}{\partial \dot{\underline{r}}_i} \left(\frac{d\underline{h}^S}{ds} \right)_{S=0} = \sum_{i=1}^N m_i \dot{\underline{r}}_i \cdot (\underline{r}_i \times \underline{e}_z)$$

$$= \underline{e}_z \cdot \sum_{i=1}^N (m_i \dot{\underline{r}}_i \times \underline{r}_i) = - \underline{e}_z \cdot \sum_{i=1}^N (\underline{r}_i \times m_i \dot{\underline{r}}_i) = - \underline{L}_z$$

Drehimpuls $\underline{L}$

Rotationsinvarianz $\Rightarrow$ Drehimpuls erhalten

• Alternative Betrachtungsweise:

Wähle $q_1 = \varphi$ als generalisierte Koordinate: $\underline{r}_i = \underline{r}_i(\varphi, q_2, \dots, q_k, t)$

mit $\frac{\partial}{\partial q_1} \underline{r}_i = \left(\frac{d}{ds} \underline{h}^i(\underline{r}_i) \right)_{s=0} = \underline{r}_i \times \underline{e}_z$ (infinitesimale Drehung um z-Achse)

Invarianz:

Erhaltungssatz

$$\frac{\partial \mathcal{L}}{\partial q_1} = 0 \quad (\Leftrightarrow) \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = 0 \quad (\Leftrightarrow) \quad P_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \text{const}$$

$$\hookrightarrow P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial T}{\partial \dot{\varphi}} = \frac{1}{2} \sum_{i=1}^N m_i \frac{\partial}{\partial \dot{\varphi}} (\dot{\underline{r}}_i)^2 = \sum_{i=1}^N m_i \dot{\underline{r}}_i \frac{\partial \dot{\underline{r}}_i}{\partial \dot{\varphi}} = \sum_{i=1}^N m_i \dot{\underline{r}}_i (\underline{r}_i \times \underline{e}_z) = \sum_{i=1}^N m_i \dot{\underline{r}}_i \underline{r}_i \times \underline{e}_z = \frac{\partial \underline{r}_i}{\partial \dot{\varphi}} \times \underline{e}_z$$

$$= -\underline{e}_z \sum_{i=1}^N (\underline{r}_i \times m_i \dot{\underline{r}}_i) = -\underline{L}_z \quad (\text{z-Komponente des Drehimpulses})$$

Bsp: N Teilchen mit ~~innerer~~ paarweiser Wechselwirkung, die nur vom

Abstand abhängt: $V(\underline{r}_1, \dots, \underline{r}_N) = V(\underline{r}_{12}, \dots, \underline{r}_{N-1,N})$ mit $\underline{r}_{ij} = |\underline{r}_i - \underline{r}_j|$

Rotationen nur ausserhalb gegen Drehung um alle Achsen (effektiv: Winkel φ)

$$\frac{\partial V}{\partial \varphi} = \sum_{i,j=1}^N \frac{\partial V}{\partial r_{ij}} \cdot \frac{\partial r_{ij}}{\partial \varphi} \stackrel{!}{=} 0$$

$$\hookrightarrow = \frac{\partial}{\partial \varphi} [(\underline{r}_i - \underline{r}_j)(\underline{r}_i - \underline{r}_j)]^{\frac{1}{2}} = \frac{1}{r_{ij}} (\underline{r}_i - \underline{r}_j) \frac{\partial}{\partial \varphi} (\underline{r}_i - \underline{r}_j) = \frac{\underline{r}_i - \underline{r}_j}{r_{ij}} \left(\frac{\partial \underline{r}_i}{\partial \varphi} - \frac{\partial \underline{r}_j}{\partial \varphi} \right)$$

$\underline{r}_i \times \underline{e}_z \quad \underline{r}_j \times \underline{e}_z$

$$= \frac{1}{r_{ij}} (\underline{r}_i - \underline{r}_j) [(\underline{r}_i - \underline{r}_j) \times \underline{e}_z] = \frac{1}{r_{ij}} \underline{e}_z [(\underline{r}_i - \underline{r}_j) \times (\underline{r}_i \times \underline{r}_j)] = 0$$

$= 0$

$\Rightarrow$ Drehimpuls $\underline{L}$ ist Erhaltungsgröße.

Zeitliche Translationsinvarianz:

(i) Zwangsbedingungen nicht explizit t -abhängig

$$\Rightarrow r_i = r_i(q_1, \dots, q_f), \quad \frac{\partial r_i}{\partial t} = 0 \Rightarrow \dot{r}_i = \sum_{j=1}^f \frac{\partial r_i}{\partial q_j} \dot{q}_j$$

$$(ii) \frac{\partial L}{\partial t} = 0$$

• Beobachtung: $\sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j \stackrel{!}{=} \sum_{j=1}^f \left(\frac{\partial T}{\partial \dot{q}_j} \right) \dot{q}_j = \left(\sum_{j=1}^f m_j \dot{q}_j \dot{q}_j \right) \stackrel{!}{=} 2T \quad (*)$

$L = T(\dot{q}) - V(q)$
 $\frac{1}{2} \sum_{j=1}^f m_j \dot{q}_j^2$
 Achtung! (Euler-Satz)
 genau! Siehe 3.2a
 gilt nicht immer!

$$\frac{d}{dt} L = \sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial q_j} \dot{q}_j \right) + \frac{\partial L}{\partial t}$$

$$= \sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j \right)$$

$$= \frac{d}{dt} \left(\sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) \stackrel{(*)}{=} 2 \frac{dT}{dt}$$

$$\Rightarrow 0 = \frac{d}{dt} (2T - L) \stackrel{L=T-V}{=} \frac{d}{dt} (T + V) \Rightarrow T + V = \text{const.}$$

Zeittranslationsinvarianz $\Rightarrow$ Energieerhaltung

Stetigkeitsbedingung $\frac{\partial L}{\partial t} = 0$
 Zwangsbedingung

$$\Rightarrow E = T + V = \text{const.}$$

Kinetische Energie: Nachtrag

$$T = \frac{1}{2} \sum_i m_i \dot{r}_i^2 = \frac{1}{2} \sum_{j,k} T_{jk} \dot{q}_j \dot{q}_k$$

mit $T_{jk} = \sum_{i=1}^N m_i \left(\frac{\partial r_i}{\partial q_j} \right) \left(\frac{\partial r_i}{\partial q_k} \right)$ abhängig von $q_1, \dots, q_k$!

T ist homogen quadratische Funktion in $\dot{q}_1, \dots, \dot{q}_f$:

$$T(\lambda \dot{q}_1, \dots, \lambda \dot{q}_f) = \lambda^2 T(\dot{q}_1, \dots, \dot{q}_f)$$

$$\frac{\partial}{\partial \lambda} \Big|_{\lambda=1} \Rightarrow \sum_k \frac{\partial T}{\partial (\lambda \dot{q}_k)} \frac{\partial (\lambda \dot{q}_k)}{\partial \lambda} \Big|_{\lambda=1} = 2 \lambda T(\dot{q}_1, \dots, \dot{q}_f) \Big|_{\lambda=1}$$

$\underbrace{\qquad\qquad\qquad}_{\dot{q}_k}$

$$\sum_{k=1}^f \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

Bsp. I in Kugelkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & m r^2 & 0 \\ 0 & 0 & m r^2 \sin^2 \vartheta \end{pmatrix}$

I in Zylinderkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & m r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

da V unabhängig von $\dot{q}_1, \dots, \dot{q}_f$ ist gilt: $\sum_{k=1}^f \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = 2T$

• Betrachte: $\bar{L} = (q_j, t, \frac{dq_j}{d\tau}, \frac{dt}{d\tau}) = L(q_j, \frac{1}{\frac{dt}{d\tau}} \frac{dq_j}{d\tau}, t) \frac{dt}{d\tau}$

also $t \rightarrow$ q-artige Variable: $q_j = q_j(\tau)$, $t = t(\tau)$, $\dot{t} = \frac{dt}{d\tau} = \dot{q}_{t+1}$

$\bar{L}$ sei invariant unter Zeittrafo: $L^s(q, t) = (q, t+s)$

$\Rightarrow$ (i) Hamiltonsches Prinzip für $\bar{L}$

$$0 = \int_{\tau_1}^{\tau_2} d\tau \bar{L} = \int_{t_1}^{t_2} dt L \Leftrightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

(ii) Noether-Theorem für $\bar{L}$:

Integraler Beweis: $I = \sum_{j=1}^{f+1} \frac{\partial \bar{L}}{\partial \dot{q}_j} \left(\frac{d}{ds} h^s(q_1, \dots, q_f, \dot{q}_{f+1}) \right) \Big|_{s=0} = \frac{\partial \bar{L}}{\partial \dot{q}_{f+1}}$

$$\begin{aligned} &= \frac{\partial \bar{L}}{\partial \dot{q}_{f+1}} = L + \sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \frac{\partial}{\partial \left(\frac{dt}{d\tau} \right)} \left(\frac{1}{\frac{dt}{d\tau}} \right) \frac{dt}{d\tau} \\ &= - \frac{1}{\left(\frac{dt}{d\tau} \right)^2} \frac{\partial L}{\partial \tau} \end{aligned}$$

$$= L - \underbrace{\sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j}_{=2T} = T - V - 2T = -(T+V)$$