4.3 Rotational invariance

Rotation w.r.t. z axis: $\underline{h}^s: \underline{r}_i \to \underline{r}_i' = \underline{h}^{s=0}(\underline{r}_i) + s \frac{d}{ds} \underline{h}^s(\underline{r}_i)\Big|_{s=0}$
(infinitesimal)

$$= \underline{r}_i + s\,(\underline{r}_i \times \underline{e}_z)$$

- invariant $L(\underline{r}_i, \dot{\underline{r}}_i) = L(\underline{r}_i', \dot{\underline{r}}_i)$ (rotation is an orthogonal transformation ⇒ only distance dependent)

$$\Rightarrow \left(\frac{dL}{ds}\right)_{s=0} = -\left(\frac{dV}{ds}\right)_{s=0} = -\sum_{i=1}^{N} \underbrace{\nabla_{\underline{r}_i} V}_{-\underline{F}_i} \cdot \underbrace{\frac{d\underline{r}_i'}{ds}\Big|_{s=0}}_{\underline{r}_i \times \underline{e}_z} = -\underline{e}_z \underbrace{\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i}_{\text{total torque}} \overset{!}{=} 0$$

Integral of motion

$$\Rightarrow I = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \left(\frac{d\underline{h}^s}{ds}\right)_{s=0} = \ldots = -\underline{e}_z \sum_{i=1}^{N} (\underline{r}_i \times m_i \dot{\underline{r}}_i) = -l_z$$

rotational invariance ⇒ conservation of angular momentum

- $\frac{\partial L}{\partial \varphi} = 0$ cyclic variable $\Leftrightarrow \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{\varphi}}\right)}_{P_\varphi} = 0 \Leftrightarrow P_\varphi = \text{const}$

4.4 Temporally translational invariance

(i) no explicitly time-dependent constraint: $\underline{r}_i = \underline{r}_i(q_1, \ldots, q_f)$

$$\frac{\partial \underline{r}_i}{\partial t} = 0 \Rightarrow \dot{\underline{r}}_i = \sum_{j=1}^{f} \frac{\partial \underline{r}_i}{\partial q_j} \dot{q}_j$$

(ii) $\frac{\partial L}{\partial t} = 0$ (scleronomic constraints)

$$\frac{d}{dt} \underset{L = T(\dot{q}) - V(q)}{L} \overset{(1)}{=} \ldots = 2\frac{dT}{dt} \Rightarrow 0 = \frac{d}{dt}(2T - L) = \frac{d}{dt}(T+V)$$

$$\Rightarrow T + V = \text{const}$$

temporally translational invariance ⇒ conservation of energy

4.4 Zeitliche Translationsinvarianz

Zeitliche Translationsinvarianz:

(i) Zwangsbedingungen nicht explizit t-abhängig

$$\Rightarrow \underline{r}_i = \underline{r}_i(q_1, \dots, q_f), \quad \frac{\partial \underline{r}_i}{\partial t} = 0 \Rightarrow \dot{\underline{r}}_i = \sum_{j=1}^{f} \frac{\partial \underline{r}_i}{\partial q_j} \dot{q}_j$$

(ii) $\frac{\partial L}{\partial t} = 0$

• Beobachtung:

$$\sum_{j=1}^{f} \left(\frac{\partial L}{\partial \dot{q}_j}\right) \dot{q}_j \underset{\uparrow}{=} \sum_{j=1}^{f} \left(\frac{\partial T}{\partial \dot{q}_j}\right) \dot{q}_j \left(= \underbrace{\sum_{j=1}^{f} m_j \dot{q}_j \dot{q}_j}_{\text{Achtung! siehe 32a}} \right) \underset{\uparrow}{=} 2T \quad (*)$$

$$\begin{pmatrix} L = T(\dot{q}) - V(q) \\ \frac{1}{2} \sum_{j=1}^{f} m_j \dot{q}_j^2 \end{pmatrix}$$

genauer! ← gilt nicht immer!

(Eulerscher Satz)

•
$$\frac{d}{dt} L = \sum_{j=1}^{f} \left(\frac{\partial L}{\partial \dot{q}_j} \underbrace{\ddot{q}_j}_{\frac{d\dot{q}_j}{dt}} + \underbrace{\frac{\partial L}{\partial q_j}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)} \dot{q}_j \right) + \underbrace{\frac{\partial L}{\partial t}}_{\overset{(ii)}{=} 0}$$

$$= \sum_{j=1}^{f} \left(\frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \dot{q}_j \right)$$

$$= \frac{d}{dt}\left(\sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) \overset{(*)}{=} 2 \frac{dT}{dt}$$

$$\Rightarrow \quad 0 = \frac{d}{dt}(2T - L) \underset{L = T - V}{=} \frac{d}{dt}(T + V) \Rightarrow T + V = \text{constant}$$

Zeittranslationsinvarianz $\Rightarrow$ Energieerhaltung

skleronome Zwangsbedingung, $\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad E = T + V = \text{const.}$

Kinetische Energie: Nachtrag

$$T = \frac{1}{2} \sum_i m_i \dot{\underline{r}}_i^2 = \frac{1}{2} \sum_{jk} T_{jk} \dot{q}_j \dot{q}_k$$

mit $T_{jk} = \sum_{i=1}^{N} m_i \left(\frac{\partial \underline{r}_i}{\partial q_j}\right) \left(\frac{\partial \underline{r}_i}{\partial q_k}\right)$ abhängig von $q_1, \dots, q_f$!

T ist homogen quadratische Funktion in $\dot{q}_1, \dots, \dot{q}_f$:

$$T(\lambda \dot{q}_1, \dots, \lambda \dot{q}_f) = \lambda^2 T(\dot{q}_1, \dots, \dot{q}_f)$$

$\frac{\partial}{\partial \lambda}\Big|_{\lambda=1}$

$$\Rightarrow \sum_k \frac{\partial T}{\partial(\lambda \dot{q}_k)} \underbrace{\frac{\partial(\lambda \dot{q}_k)}{\partial \lambda}}_{\dot{q}_k}\Bigg|_{\lambda=1} = 2\lambda T(\dot{q}_1, \dots, \dot{q}_f)\Big|_{\lambda=1}$$

$$\sum_{k=1}^{f} \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

Bsp. $\underline{T}$ in Kugelkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \sin^2\vartheta \end{pmatrix}$

$\underline{T}$ in Zylinderkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

da V unabhängig von $\dot{q}_1, \dots, \dot{q}_f$ ist gilt: $\sum_{k=1}^{f} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \dot{q}_k = 2T$

- Betrachte: $\bar{L}(\underbrace{q_j, t}_{q_1,\dots,q_f,\ q_{f+1}}, \frac{dq_j}{d\tau}, \frac{dt}{d\tau}) := L(q_j, \frac{1}{\frac{dt}{d\tau}}\frac{dq_j}{d\tau}, t)\frac{dt}{d\tau}$

also $t \to$ q-artige Variable: $q_j = q_j(\tau)$, $t = t(\tau)$, $t = q_{f+1}$, $\frac{dt}{d\tau} = \dot{q}_{f+1}$

L sei invariant unter Zeittrafo: $h^s(q,t) = (q, t+s)$

$\Rightarrow$ (i) Hamiltonsches Prinzip für $\bar{L}$

$$0 = \delta\int_{\tau_1}^{\tau_2} d\tau\, \bar{L} = \delta\int_{t_1}^{t_2} dt\, L \Leftrightarrow \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

(ii) Noether-Theorem für $\bar{L}$:

Integral der Bewegung: $$I = \sum_{j=1}^{f+1}\frac{\partial \bar{L}}{\partial \dot{q}_j}\underbrace{\left(\frac{d}{ds}h_i^s(q_1,\dots,q_f,\overset{t}{q_{f+1}})\right)\Big|_{s=0}}_{(0,\dots,0,1)} = \frac{\partial \bar{L}}{\partial \dot{q}_{f+1}}$$

$$\underset{\substack{q_{f+1}=t(\tau)\\ \dot{q}_{f+1}=\frac{dt}{d\tau}}}{=}\frac{\partial \bar{L}}{\partial\left(\frac{dt}{d\tau}\right)} = L + \sum_{j=1}^{f}\frac{\partial L}{\partial \dot{q}_j}\underbrace{\frac{\partial}{\partial\left(\frac{dt}{d\tau}\right)}\left(\frac{1}{\frac{dt}{d\tau}}\frac{dq_j}{d\tau}\right)}_{= -\frac{1}{\left(\frac{dt}{d\tau}\right)^2}\frac{dq_j}{d\tau}}\frac{dt}{d\tau}$$

$$= L - \underbrace{\sum_{j=1}^{f}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j}_{=2T} = T - V - 2T = -(T+V)$$

4.5 Das Zweikörperproblem

Idee: Herleitung der Bewegungsgleichung über Reduzierung der Freiheitsgrade / Identifizierung der Integrale der Bewegung:

Allgemein: f Freiheitsgrade $\Rightarrow$ f DGL 2. Ordnung $\Rightarrow$ $2f$ Integrationskonstanten (Anfangsbedingungen)

$\Rightarrow$ max. $2f$ Integrale der Bewegung

(dann wäre Problem gelöst: $I_\nu(q, \dot{q}_f) = c_\nu$, $\nu = 1, \dots, f$)

Hier: **Zweikörperproblem**: 2 Massen m_1, m_2 mit innerer Wechselwirkung

$V(|\underline{r}_1 - \underline{r}_2|)$: **Zentralpotential**

(Sonne-Erde, Erde-Mond)

$\Rightarrow f = 6 \Rightarrow 12$ Integrale der Bewegung möglich

(i) $V(|\underline{r}_1 - \underline{r}_2|)$ translationsinvariant

$\Rightarrow \underline{P} = \underline{P}_1 + \underline{P}_2 = \text{const}$ (Impuls $\underline{P} = M\dot{\underline{R}}$)

$\Rightarrow$ Schwerpunkt: $\underline{R} = \frac{1}{M}\underline{P}t + \underline{R}_0$, $M = m_1 + m_2$

$\Rightarrow$ Schwerpunkt bewegt sich geradlinig und gleichförmig

$\Rightarrow$ 6 Integrationskonstanten $\underline{P}, \underline{R}_0$

(ii) $V(|\underline{r}_1 - \underline{r}_2|)$ rotationsinvariant

$\Rightarrow \underline{\ell} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2 = \text{const}$ (Drehimpuls)

$\Rightarrow$ 3 Integrationskonstanten $\underline{\ell}$

(iii) zeitliche Translationsinvarianz, konservative Kraft

$\Rightarrow E = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 + V(|\underline{r}_1 - \underline{r}_2|) = \text{const}$ (Energie)

$\Rightarrow$ 1 Integrationskonstante E

$\Rightarrow$ 10 Integrale der Bewegung identifiziert

$\Rightarrow$ Es bleiben $12 - 10 = 2$ Integrationskonstanten (Nullpunkt von Zeit- & Winkelskala)

4.5.1 Impuls- & Drehimpulserhaltung

Idee: Lagrange-Formalismus: $L = T - V = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 - V(|\underline{r}_1 - \underline{r}_2|)$

Verallgemeinerte Koordinaten:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underline{R} = \frac{1}{M}(m_1 \underline{r}_1 + m_2 \underline{r}_2)$$ Schwerpunktskoordinate

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \end{pmatrix} = \underline{r} = \underline{r}_1 - \underline{r}_2$$ Relativkoordinate

$\hookrightarrow \underline{r}_1 = \underline{R} + \frac{m_2}{M}\underline{r}$, $\underline{r}_2 = \underline{R} - \frac{m_1}{M}\underline{r}$

$\dot{\underline{r}}_1 = \dot{\underline{R}} + \frac{m_2}{M}\dot{\underline{r}}$, $\dot{\underline{r}}_2 = \dot{\underline{R}} - \frac{m_1}{M}\dot{\underline{r}}$

$\Rightarrow L = \frac{M}{2}\dot{\underline{R}}^2 + \frac{1}{2} m \dot{\underline{r}}^2 - V(r)$, $r = |\underline{r}|$ Abstand, $m = \frac{m_1 m_2}{m_1 + m_2}$ reduzierte Masse

- $\frac{\partial L}{\partial R_k} = 0 \Rightarrow \underline{R}$ ist zyklische Variable $\Rightarrow \frac{\partial L}{\partial \dot{R}_k} = M\dot{R}_k = P_k = \text{const}$

  $\Rightarrow \underline{R} = \frac{t}{M}\underline{P} + \underline{R}_0$

- Schwerpunktsystem als Inertialsystem! OBdA $\underline{R} = \dot{\underline{R}} = 0$

$\Rightarrow L = \frac{1}{2} m \dot{\underline{r}}^2 - V(r)$, $\underline{r}_1 = \frac{m_2}{M}\underline{r}$, $\underline{r}_2 = -\frac{m_1}{M}\underline{r}$

$\dot{\underline{r}}_1 = \frac{m_2}{M}\dot{\underline{r}}$, $\dot{\underline{r}}_2 = -\frac{m_1}{M}\dot{\underline{r}}$

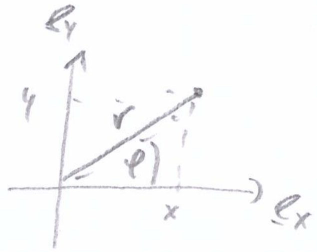
Drehimpuls: $\underline{\ell} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2$

$$= m_1 \frac{m_2}{M}\underline{r} \times \frac{m_2}{M}\dot{\underline{r}} + m_2\left(-\frac{m_1}{M}\right)\underline{r} \times \left(-\frac{m_1}{M}\right)\dot{\underline{r}}$$

$$= \underbrace{\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right)}_{\frac{(m_1+m_2)\, m_1 m_2}{M^2} = \frac{m_1 m_2}{M} = m} \underline{r} \times \dot{\underline{r}} = m\, \underline{r} \times \dot{\underline{r}} = \text{const}$$

$\Rightarrow \underline{\ell} \cdot \underline{r} = 0$, $\underline{\ell} \cdot \dot{\underline{r}} = 0 \Rightarrow \underline{r}$ und $\dot{\underline{r}}$ in Ebene senkrecht zu $\underline{\ell}$ (im Schwerpunktsystem)

$\Rightarrow$ 2D Bewegung! $\Rightarrow$ Polarkoordinaten
(oBdA $\underline{\ell} \parallel \underline{e}_z$)



$x = r\cos\varphi \Rightarrow \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$

$y = r\sin\varphi \Rightarrow \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$

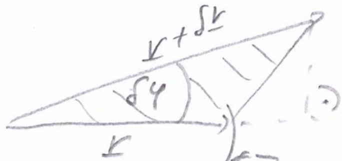
$$\Rightarrow \dot{\underline{r}}^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2(\cos^2\varphi + \sin^2\varphi) + r^2\dot{\varphi}^2(\sin^2\varphi + \cos^2\varphi) = \dot{r}^2 + r^2\dot{\varphi}^2$$

$$\Rightarrow L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

- $\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi$ ist zyklisch $\Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = \underset{\uparrow \atop m(x\dot{y} - y\dot{x}) = m r^2\dot{\varphi}}{\ell_z} \overset{\ell_x = \ell_y = 0}{=} \ell = \text{const}$

- **2. Keplersches Gesetz**: Flächensatz (geometrische Interpretation von $m r^2 \dot{\varphi} = \ell = \text{const}$)

Der Radiusvektor $\underline{r}$ überstreicht in gleichen Zeiten gleiche Flächen.
(Flächengeschwindigkeit $\frac{dF}{dt} = \text{const}$)



$\} \; |\underline{r} + \delta\underline{r}| \sin\delta\varphi$

$$\delta F = \frac{1}{2}|\underline{r}|\,|\underline{r} + \delta\underline{r}| \sin\delta\varphi \underset{\uparrow \atop {\delta r, \delta\varphi \to 0 \atop \sin\delta\varphi \approx \delta\varphi}}{\approx} \frac{1}{2}|\underline{r}|\,|\underline{r}|\,\delta\varphi$$

$$\Rightarrow \frac{dF}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2} r^2 \dot{\varphi} \underset{\uparrow \atop m r^2\dot{\varphi} = \ell}{=} \frac{\ell}{2m} = \text{const!}$$