4.5 Two-body problem

2 (point) masses in 3D $\Rightarrow f = 6$ degrees of freedom

$\Rightarrow 2f = 12$ integration constants / initial values

potential: $V(\underline{r}_1, \underline{r}_2) = V(|\underline{r}_1 - \underline{r}_2|)$ distance-dependent

(i) translational invariance $\Rightarrow$ conservation of momentum $\underline{P} = M\dot{\underline{R}} = \text{const}$

$\Rightarrow \underline{R}(t) = \frac{\underline{P}}{M} t + \underline{R}_0$

$\Rightarrow$ 3 initial values $\underline{P}$

3 — " — $\underline{R}_0$

(ii) rotational invariance $\Rightarrow$ conservation of angular momentum $\underline{l} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2 = \text{const}$

$\Rightarrow$ 3 initial conditions $\underline{l}$

(iii) temporally translational invariance $\Rightarrow$ conservation of energy

$$E = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 + V(|\underline{r}_1 - \underline{r}_2|) = \text{const}$$

$\Rightarrow$ 1 initial value E

$\Rightarrow$ 10 of 12 initial values given by system setup

coordinate of centre of mass: $\underline{R} = \frac{1}{M}(m_1 \underline{r}_1 + m_2 \underline{r}_2)$

relative coordinate: $\underline{r} = \underline{r}_1 - \underline{r}_2$

$\Rightarrow \underline{r}_1 = \underline{R} + \frac{m_2}{M}\underline{r}$, $\quad \underline{r}_2 = \underline{R} - \frac{m_1}{M}\underline{r}$

$\Rightarrow$ Lagrangian: $L = \frac{M}{2}\dot{\underline{R}}^2 + \frac{m}{2}\dot{\underline{r}}^2 - V(r)$, $\quad m = \frac{m_1 m_2}{m_1 + m_2}$ reduced mass

$\Rightarrow \underline{R}$ is acyclic variable $\Leftrightarrow \frac{\partial L}{\partial \dot{R}_k} = \text{const} = P_k \Rightarrow \underline{R}(t) = \frac{\underline{P}}{M} t + \underline{R}_0$

$\frac{\partial L}{\partial R_k} = 0$

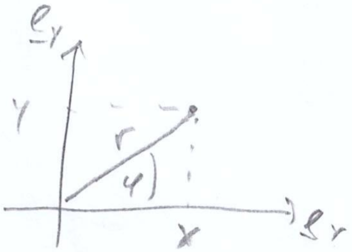
w.l.g. $\underline{P} = 0 = \underline{R}_0 \Rightarrow \underline{R} = 0$

angular momentum $\underline{l} = \ldots = m\,\underline{r} \times \dot{\underline{r}}$

$\Rightarrow \underline{l} \cdot \underline{r} = 0\,, \quad \underline{l} \cdot \dot{\underline{r}} = 0$ (motion in a plane)

$\Rightarrow$ effective 2D problem (w.l.g. $\underline{l} \parallel \underline{e}_z$

$\Rightarrow$ polar coordinates

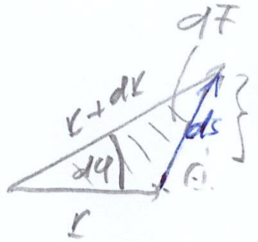


$$\left.\begin{aligned} x &= r\cos\varphi \Rightarrow \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ y &= r\sin\varphi \Rightarrow \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{aligned}\right\} \Rightarrow \dot{\underline{r}}^2 = \dot{r}^2 + r^2\dot{\varphi}^2$$

$$\Rightarrow L = \frac{m}{2}\left(\dot{r} + r^2\dot{\varphi}^2\right) - V(r)$$

$$\Rightarrow \varphi \text{ is a cyclic variable} \Leftrightarrow \frac{\partial L}{\partial \varphi} = 0 \qquad \frac{\partial L}{\partial \dot{\varphi}} = \text{const} = p_\varphi = m r^2 \dot{\varphi} = \overset{m(x\dot{y} - \dot{x}y)}{l_z} = \underset{\underline{l} \parallel \underline{e}_z}{l}$$

$\Rightarrow$ Kepler's 2^{nd} law: A line joining a planet and the sun sweeps out equal areas during equal intervals of time.



$$dF = \frac{1}{2}|r|\,|r + dr| \sin d\varphi$$

$$\xrightarrow[dt \to 0]{dr \to 0} \frac{dF}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2} r^2 \frac{l}{m r^2} = \frac{l}{2m} = \text{const}$$

alternative: $dF = \frac{1}{2}\left|\underline{r} \times d\underline{s}\right| = \frac{1}{2}\left|\underline{r} \times \dot{\underline{r}}\,dt\right| \Rightarrow \frac{dF}{dt} = \frac{1}{2}\left|\underline{r} \times \dot{\underline{r}}\right|$

$$= \frac{1}{2m}\left|\underline{r} \times m\dot{\underline{r}}\right|$$

$$= \frac{1}{2m}\, l$$

4.5.2 Energieerhaltung und Bahngleichung

Lagrange-Gleichung 2. Art für r: $\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot r}}_{m\dot r}\Big) - \underbrace{\frac{\partial L}{\partial r}}_{mr\dot\varphi^2 - V'(r)} = 0$

$$\Rightarrow \quad m\ddot r - \underbrace{mr\dot\varphi^2}_{\text{Zentrifugalkraft}} + V'(r) = 0 \qquad \dot\varphi = \frac{\ell}{mr^2}$$

$$\Rightarrow \quad m\ddot r - \frac{\ell^2}{mr^2} + V'(r) = 0 \qquad \Big|\cdot \dot r$$

$\uparrow$ $\dot\varphi = \frac{\ell}{mr^2}$

$$\underbrace{m\dot r\ddot r}_{\frac{d}{dt}\left(\frac{m}{2}\dot r^2\right)} + \frac{\ell^2}{m}\underbrace{\left(-\frac{\dot r}{r^3}\right)}_{\frac{d}{dt}\frac{1}{2r^2}} + \underbrace{\dot r V'(r)}_{\frac{d}{dt}V(r)}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{m}{2}\dot r^2 + \frac{\ell^2}{2mr^2} + V(r)\right) = 0 \qquad \int dt$$

$$\underbrace{\frac{m}{2}\dot r^2 + \frac{\ell^2}{2mr^2}}_{\frac{m}{2}\left(\dot r^2 + \frac{m r^4\dot\varphi^2}{mr^2}\right) = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2\right) = T} + V(r) = E = \text{const} \qquad E = T + V$$

Alternativ: $E = \frac{m}{2}\dot r^2 + \tilde V(r)$ mit effektivem Radialpotential

$$\tilde V(r) = V(r) + \frac{\ell^2}{2mr^2}$$

$$\Rightarrow \quad \dot r = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - \tilde V(r)\right)}$$

Trennung der Variablen:

$$\int_{t_0}^{t} dt' = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left(E - \tilde V(r)\right)}} \quad \Rightarrow \quad t = t(r) \Rightarrow \text{Umkehrung: } r = r(t)$$

$$\Rightarrow \quad \varphi(t) \text{ aus } \dot\varphi = \frac{d\varphi}{dt} = \frac{\ell}{mr^2(t)}: \quad \int_{\varphi_0}^{\varphi} d\varphi' = \int_{t_0}^{t} dt' \frac{\ell}{mr^2(t')} \Rightarrow \varphi = \varphi(t)$$

Bahngleichung: $r = r(\varphi)$, $\varphi = \varphi(r)$

$$\frac{dr}{d\varphi} = \frac{\dot r}{\dot\varphi} = \frac{m r^2}{\ell}\sqrt{\frac{2}{m}\left(E - \tilde V(r)\right)}$$

$$\int d\varphi = \int \frac{dr}{r^2\sqrt{\frac{2m}{\ell^2}\left(E - \tilde V(r)\right)}}$$

4.5.3 Planetenbewegung & Keplersche Gesetze

Nun: Gravitationspotential: $V(r) = -\frac{\gamma m_1 m_2}{r}$

$$\Rightarrow \tilde V(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2}, \quad k = \gamma m_1 m_2 > 0$$

$r \to 0$: $\tilde V(r) \sim \frac{\ell^2}{2mr^2} \to \infty$

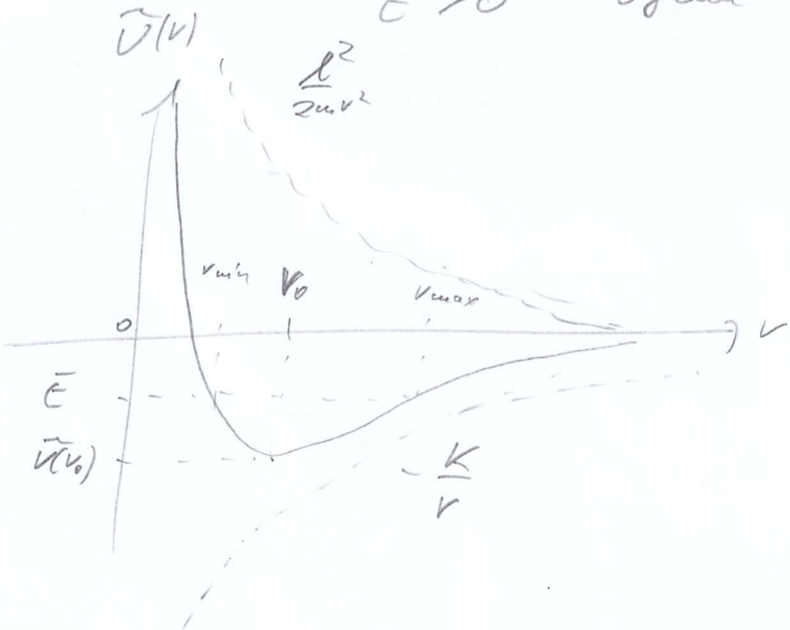
$r \to \infty$: $\tilde V(r) \sim -\frac{k}{r} \to 0$

Minimum: $\frac{d\tilde V}{dr} = \frac{k}{r^2} - \frac{2\ell^2}{2mr^3} \overset{!}{=} 0 \Rightarrow r_0 = \frac{\ell^2}{mk} \Rightarrow \tilde V(r_0) = -\frac{mk^2}{2\ell^2}$

Wegen $\frac{m}{2}\dot r^2 = E - \tilde V(r)$ ist eine Bewegung nur für $\tilde V(r) < E$ möglich

$\Rightarrow -\frac{mk^2}{2\ell^2} < E < 0$ geschlossene Bahn

$E > 0$ offene Bahn



Für $-\frac{mk^2}{2\ell^2} < E < 0$ sind die Wendepunkte durch

$$\tilde{V}(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} = E \quad \text{bestimmt } (E<0)$$

$$\Rightarrow r_{min/max} = \frac{1}{2|E|}\left(k \pm \sqrt{k^2 - \frac{2\ell^2|E|}{m}}\right)$$

$$\left[\quad Er^2 + kr - \frac{\ell^2}{2m} = 0\right.$$

$$r^2 - \frac{k}{|E|}r + \frac{\ell^2}{2m|E|} = 0$$

$$\hookrightarrow r_{min/max} = \frac{1}{2}\frac{k}{|E|} \mp \sqrt{\frac{k^2}{|E|^2} - 4\frac{\ell^2}{2m}}$$

$$\left. = \frac{1}{2}\frac{1}{|E|}\left(k \mp \sqrt{k^2 - \frac{2\ell^2|E|}{m}}\right)\quad\right]$$

$E>0$: nur 1 Wendepunkt: $r_{min} = \frac{1}{2E}\left(\sqrt{k^2 + \frac{2\ell^2 E}{m}} - k\right)$

Bahngleichung: $$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}(E - \tilde{V}(r'))}}$$

$$= \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}E + \frac{2mk}{\ell^2 r'} - \frac{1}{r'^2}}}$$

$$= \ldots = \int_{\vartheta_0}^{\vartheta} d\vartheta' \quad \text{mit } \cos\vartheta = \frac{1}{\sqrt{D}}\left(\frac{1}{r} - \frac{mk}{\ell^2}\right)$$