4.5.2 Conservation of energy & trajectory (around the sun / star)

Euler-Lagrange equation: $\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = 0$

$$L = \frac{1}{2}m\left(\dot r^2 + r^2\dot\varphi^2\right) - V(r)$$

$$\Rightarrow \quad m\ddot r - mr\dot\varphi^2 + \frac{\partial V}{\partial r} = 0 \quad , \quad \dot\varphi = \frac{\ell}{mr^2}$$

$$\Rightarrow \dots \Rightarrow \quad E = \frac{m}{2}\dot r^2 + \underbrace{\frac{\ell^2}{2mr^2} + V(r)}_{=:\tilde V(r)} = \text{const}$$

centrifugal potential $\frac{\ell^2}{2mr^2} \Rightarrow F_Z = \frac{\ell^2}{2mr^3}\frac{\vec r}{r}$

$$\Rightarrow \quad t(r) - t_0 = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left(E - \tilde V(r')\right)}} \quad \Rightarrow t(r) \text{ or } r(t)$$

$$\varphi(t) - \varphi_0 = \int_{t_0}^{t} dt' \frac{\ell}{mr^2(t')} \quad \Rightarrow \varphi(t)$$

$$\frac{dr}{d\varphi} = \frac{\dot r}{\dot\varphi} = \frac{mr^2}{\ell}\sqrt{\frac{2}{m}\left(E - \tilde V(r)\right)}$$

$$\Rightarrow \varphi(r) - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}\left(E - \tilde V(r')\right)}} \quad \Rightarrow \varphi(r),\ r(\varphi)$$

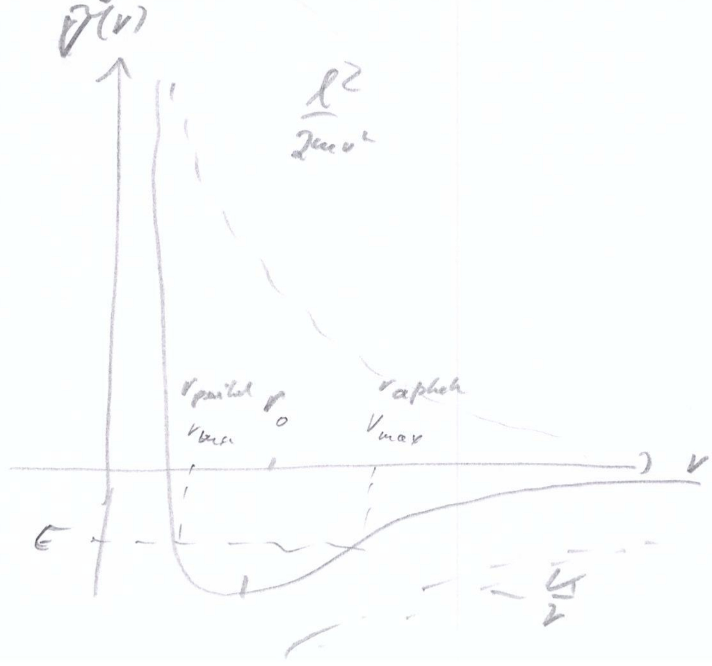
4.5.3 motion of planet & Kepler's laws

now: $V(r) = -\frac{\gamma m_1 m_2}{r} = -\frac{k}{r} \Rightarrow \tilde V(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2}$ $\begin{cases} \xrightarrow{r\to 0} \frac{\ell^2}{2mr^2} \to \infty \\ \xrightarrow{r\to\infty} -\frac{k}{r} \to 0 \end{cases}$

motion possible only for $E > \tilde V(r)$ $\quad \left(\frac{1}{2}mv^2 = E - \tilde V\right)$

closed orbit $\quad -\frac{mk^2}{2\ell^2} < E < 0$

diverging path $\quad E > 0$



$$r_{min/max} = \frac{1}{2|E|}\left(k \pm \sqrt{k^2 - \frac{2\ell^2|E|}{m}}\right)$$

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}E + \frac{2mk}{\ell^2 r'} - \frac{1}{r'^2}}}$$

Lösen des Integrals:

$$\varphi-\varphi_0=\int_{r_0}^{r}\frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}\left(E-\tilde V(r')\right)}}=\int_{r_0}^{r}\frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}\left(E+\frac{k}{r'}-\frac{\ell^2}{2mr'^2}\right)}}$$

nach (i) quadratische Ergänzung und (ii) Substitution

(i) $$\frac{2m}{\ell^2}\left(E-\tilde V(r')\right)=\frac{2mE}{\ell^2}+\frac{2mk}{\ell^2 r'}-\frac{1}{r'^2}=\underbrace{-\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2}_{=-\frac{1}{r'^2}+\frac{2mk}{\ell^2 r'}-\frac{m^2k^2}{\ell^4}}\underbrace{+\frac{m^2k^2}{\ell^4}+\frac{2mE}{\ell^2}}_{=\frac{2m}{\ell^2}\left(\frac{mk^2}{2\ell^2}+E\right)}$$

$$=D-\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2$$

$$=D\left[1-\frac{1}{D}\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2\right]\quad\text{mit } D=\frac{2m}{\ell^2}\left(\frac{mk^2}{2\ell^2}+E\right)>0$$

(ii) Substitution: $\cos\vartheta'=\frac{1}{\sqrt{D}}\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)\Rightarrow -\sin\vartheta'\,d\vartheta'=-\frac{dr'}{\sqrt{D}\,r'^2}$

$\frac{d}{dr'}\cos\vartheta'(r)=\frac{d}{dr'}(\dots)$

$$\Rightarrow\varphi-\varphi_0=\int_{r_0}^{r}\frac{dr'}{r'^2}\frac{1}{\sqrt{D}}\frac{1}{\sqrt{1-\frac{1}{D}\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2}}=\int_{\vartheta_0}^{\vartheta}d\vartheta'\sin\vartheta'\underbrace{\frac{1}{\sqrt{1-\cos^2\vartheta'}}}_{\frac{1}{\sin\vartheta'}}=\int_{\vartheta_0}^{\vartheta}d\vartheta'$$

$$=\vartheta-\vartheta_0=\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r}-\frac{mk}{\ell^2}\right)\right]-\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r_0}-\frac{mk}{\ell^2}\right)\right]$$

r_0,φ_0 Integrationskonstanten!

etwa: $\varphi_0=\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r_0}-\frac{mk}{\ell^2}\right)\right]\Rightarrow\varphi(r)=\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r}-\frac{mk}{\ell^2}\right)\right]$

$$\Rightarrow\frac{1}{r(\varphi)}=\frac{mk}{\ell^2}+\sqrt{D}\cos\varphi=\frac{mk}{\ell^2}\left(1+\varepsilon\cos\varphi\right)$$

$$\varepsilon:=\sqrt{D}\frac{\ell^2}{mk}=\sqrt{1+\frac{2E\ell^2}{mk^2}}\geq 0$$

$$\Leftrightarrow\quad r(\varphi)=\frac{\frac{\ell^2}{mk}}{1+\varepsilon\cos\varphi}\qquad\begin{cases}\varepsilon=0: & r(\varphi)=\frac{\ell^2}{mk}=\text{const}\Rightarrow\text{Kreis}\\ 0<\varepsilon<1 & \text{Ellipse}\end{cases}$$

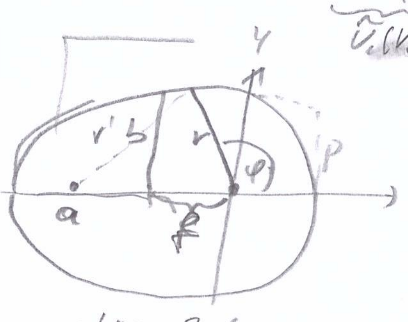
Kegelschnitte:

$\varepsilon > 1 \;\hat{=}\; E > 0$ Hyperbel — offene Bahn — $r' - r = 2a$

$\varepsilon = 1 \;\hat{=}\; E = 0$ Parabel — " —

$0 < \varepsilon < 1 \;\hat{=}\; \underbrace{-\frac{mk^2}{2l^2}}_{V_{\text{eff}}(r_0)\ \text{Minimum}} < E < 0$ Ellipse — geschlossene Bahn — $r' + r = 2a$

$\varepsilon < 1$: $\cos\varphi = \frac{x}{r}$, $r = \sqrt{x^2+y^2}$



$r' + r = 2a = \text{const}$

$$\Rightarrow \boxed{\frac{(x+e)^2}{a^2} + \frac{y^2}{b^2} = 1} \quad , \; e = \sqrt{a^2 - b^2}$$

lineare Exzentrizität

Exzentrizität

Ellipsengleichung

$\varepsilon = \frac{e}{a}$: numerische Exzentrizität

Hauptachsen: $a = \frac{l^2}{mk(1-\varepsilon^2)} = \frac{k}{2|E|}$

$(1-\varepsilon^2)\, a = \frac{l^2}{mk}$

$\Rightarrow r(\varphi) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi}$

$b = \frac{l^2}{mk\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2m|E|}}$

$\Rightarrow \frac{b^2}{a} = \frac{l^2}{mk}$

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alternative Ellipsengleichung:

$$\frac{1}{r} \left(= \frac{mk}{l^2} + \sqrt{D}\,\frac{x}{r} \right) = \frac{mk}{l^2}\left(1 + \frac{\varepsilon x}{r}\right)$$

$$\Leftrightarrow 1 = \frac{mk}{l^2}\sqrt{x^2+y^2} + \frac{mk}{l^2}\varepsilon x$$

$$\Leftrightarrow \sqrt{x^2+y^2} = \frac{l^2}{mk} - \varepsilon x$$

$$\Leftrightarrow x^2 + y^2 = \left(\frac{l^2}{mk}\right)^2 - \frac{l^2\, 2\varepsilon x}{mk} + \varepsilon^2 x^2$$

$$\Leftrightarrow y^2 = \left(\frac{l^2}{mk}\right)^2 - \frac{l^2\, 2\varepsilon}{mk}\, x + (\varepsilon^2 - 1)\, x^2$$

3. Keplersches Gesetz

$T^2 \sim a^3$ (T: Umlaufszeit, a: große Halbachse)

Ellipsenfläche $F = \pi a b$

mit $\frac{dF}{dt} = \frac{l}{2m} = \text{const}$ folgt: $F = \pi a b \int_0^T \frac{dF}{dt} dt = \frac{l}{2m} T$

$$\Rightarrow T = \frac{2m\pi ab}{l} \underset{\uparrow}{=} \frac{2m\pi a}{l} \frac{l}{\sqrt{mk}} \sqrt{a} = 2\pi \sqrt{\frac{m}{k}}\, a^{3/2}$$

$$\frac{b^2}{a} = \frac{l^2}{mk} \Rightarrow b = \frac{l}{\sqrt{mk}} \sqrt{a}$$

$\Rightarrow T^2 \sim a^3$

mit $m = \frac{m_1 m_2}{m_1 + m_2}$, $k = \gamma m_1 m_2$ folgt: $\frac{m}{k} = \frac{1}{\gamma (m_1 + m_2)}$

Für m_1 (Planet) $\ll m_2$ (Sonne): $\frac{m}{k} \approx \frac{1}{\gamma m_2} \Rightarrow T \approx 2\pi \frac{a^{3/2}}{\sqrt{\gamma m_2}}$