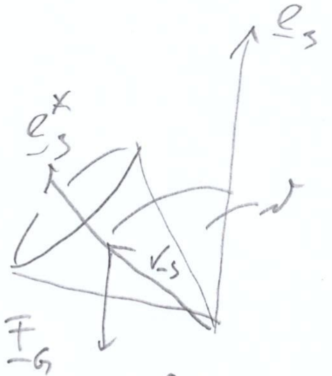
6.4 Examples:

(1) torque-free top (symmetric)

(2) symmetric top subject to gravity



$\underline{r}_s = R\,\underline{e}_3^*$ center of mass, $\underline{F}_G = -mg\,\underline{e}_3$

$\Rightarrow$ Torque: $\underline{M} = \underline{r}_s \times \underline{F}_G = -mgR\,\underline{e}_3^* \times \underline{e}_3$

$\Rightarrow \underline{M} \perp \underline{e}_3^* \wedge \underline{M} \perp \underline{e}_3$

total energy: $E = T + V = \frac{1}{2}\Theta(\omega_1^2 + \omega_2^2) + \frac{1}{2}\Theta_3\omega_3^2 + mgR\cos\vartheta$

$= \text{const}$

$\hookrightarrow \dot{u}^2 + V_{eff}(u) = 0$

motion with $-1 < u_1 \leq \cos\vartheta \leq u_2 < 1$ Nutation

(3) torque-free asymmetric: $0 < \Theta_1 < \Theta_2 < \Theta_3$

$$\left.\begin{aligned} \dot\omega_1 &= -\alpha_1\omega_2\omega_3 \\ \dot\omega_2 &= \alpha_2\omega_3\omega_1 \\ \dot\omega_3 &= -\alpha_3\omega_1\omega_2 \end{aligned}\right\}\ \alpha_i > 0 \qquad \alpha_1 = \frac{\Theta_3 - \Theta_2}{\Theta_1},\ \alpha_2 = \frac{\Theta_3 - \Theta_1}{\Theta_2},\ \alpha_3 = \frac{\Theta_2 - \Theta_1}{\Theta_3}$$

fixed points: $\underline{\omega}^* \in \left\{ \begin{pmatrix}\omega\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\omega\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\omega\end{pmatrix} \right\}$

$\Rightarrow$ linearization: $\delta\underline{\omega} = \underline{\omega} - \underline{\omega}^*$:

$$\begin{pmatrix}\delta\dot\omega_1\\ \delta\dot\omega_2\\ \delta\dot\omega_3\end{pmatrix} = \begin{pmatrix} 0 & -\alpha_1\omega_3 & -\alpha_1\omega_2 \\ \alpha_2\omega_3 & 0 & \alpha_2\omega_1 \\ -\alpha_3\omega_2 & -\alpha_3\omega_1 & 0 \end{pmatrix} \begin{pmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\end{pmatrix}$$

$\Rightarrow$ eigenvalues:

		(not unstable)
$(\omega,0,0)$	$\lambda_1 = 0,\ \lambda_{2,3} = \pm i\omega\sqrt{\alpha_2\alpha_3}$	stable / center
$(0,\omega,0)$	$\lambda_1 = 0,\ \lambda_{2,3} = \pm\omega\sqrt{\alpha_1\alpha_3}$	unstable / saddle
$(0,0,\omega)$	$\lambda_1 = 0,\ \lambda_{2,3} = \pm i\omega\sqrt{\alpha_1\alpha_2}$	stable / center

$\Rightarrow$ 1 unstable motion / rotation around axis relating to Θ_2

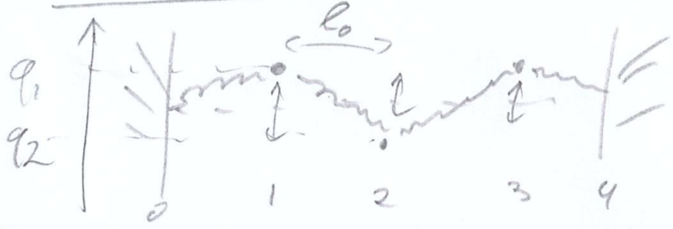
7. Kontinuumsmechanik

7.1 Kettenschwingung mit 3 Massen / 3 gekoppelte Pendel/Schwingungen

7.2 Übergang zum Kontinuum

7.3 Wellengleichung

7.1 3 gekoppelte Kettenschwingungen



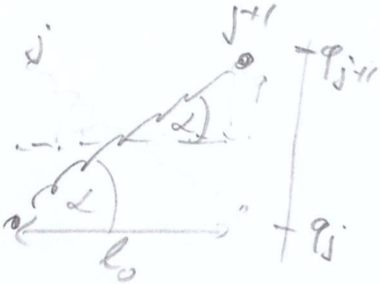
hier: transversale Bewegung

$$L = T - U$$
$$= \frac{m}{2}\sum_{i=0}^{N+1=2}\dot q_i^2 - \frac{D}{2}\sum_{i=0}^{N+1}(q_i - q_{i-1})^2$$

Bewegungsgleichung: (Herleitung über Formalismus deiner Wahl)

$$m\ddot q_j - D(q_{j-1} - 2q_j + q_{j+1}) = 0 \quad , j = 1,2,3$$

Randbedingungen $q_0(t) = q_4(t) = 0$



mit $F\sin\alpha \approx F\tan\alpha = F\,\frac{q_{j+1} - q_j}{l_0}$

$$m\ddot q_j + \underbrace{\frac{F}{l_0}}_{\hat{=}\, D}(2q_j - q_{j+1} - q_{j-1}) = 0$$

longitudinale & transversale (1d) Bewegung haben dieselbe Gleichung!

Lösung: $q_j(t) = C\, a_j \cos(\omega t - \delta)$

$$\Rightarrow \begin{aligned} &(2D - m\omega^2)a_1 - D a_2 &&= 0 \\ &-D a_1 + (2D - m\omega^2)a_2 - D a_3 &&= 0 \\ &-D a_2 + (2D - m\omega^2)a_3 &&= 0 \end{aligned}$$

$\Rightarrow$ Typ: Eigenwertgleichung (Eigenwerte ω^2 und Eigenvektoren (a_1, a_2, a_3))

$$\begin{pmatrix} 2\frac{D}{m} & -\frac{D}{m} & 0 \\ -\frac{D}{m} & 2\frac{D}{m} & -\frac{D}{m} \\ 0 & -\frac{D}{m} & 2\frac{D}{m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \cdot\ \frac{D}{m} = \omega_0^2$$

$$0 = \begin{vmatrix} 2\omega_0^2-\lambda & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2-\lambda & -\omega_0^2 \\ 0 & -\omega_0^2 & 2\omega_0^2-\lambda \end{vmatrix}$$

$$= (2\omega_0^2-\lambda)\begin{vmatrix} 2\omega_0^2-\lambda & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2-\lambda \end{vmatrix}$$

$$-(-\omega_0^2)\begin{vmatrix} -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2-\lambda \end{vmatrix}$$

$$= (2\omega_0^2-\lambda)\left[(2\omega_0^2-\lambda)^2 - \omega_0^4\right]$$

$$+\omega_0^2\left[-\omega_0^2(2\omega_0^2-\lambda)\right]$$

$$= (2\omega_0^2-\lambda)\left[(2\omega_0^2-\lambda)^2 - 2\omega_0^4\right]$$

$$= \underbrace{(2\omega_0^2-\lambda)}_{\Rightarrow \lambda_1 = 2\omega_0^2}\underbrace{\left[4\omega_0^2 - 4\omega_0^2\lambda + \lambda^2 - 2\omega_0^4\right]}_{\lambda^2 - 4\omega_0^2\lambda + 2\omega_0^4}$$

$$\Rightarrow \lambda_{2,3} = 2\omega_0^2 \pm \sqrt{4\omega_0^4 - 2\omega_0^4}$$

$$= 2\omega_0^2 \pm \sqrt{2\omega_0^4} = (2\pm\sqrt{2})\,\omega_0^2$$

$$\Rightarrow \lambda_1 = 2\omega_0^2 = 2\,\frac{D}{m}$$

$$\lambda_{2,3} = (2\pm\sqrt{2})\,\omega_0^2 = (2\pm\sqrt{2})\,\frac{D}{m}$$

Lösungen:

(i) $\omega_1^2 = \left(2 - \sqrt{2}\right)\frac{D}{m}$, $\underline{a}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

(ii) $\omega_2^2 = 2\frac{D}{m}$, $\underline{a}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ — Fundamental-Schwingungen

(iii) $\omega_3^2 = \left(2 + \sqrt{2}\right)\frac{D}{m}$, $\underline{a}^{(3)} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

Bewegungsmoden: Massen auf Sinuskurven als Einhüllende (max. Auslenkung)

(i) (ii) (iii)

ω_1 ω_2 ω_3

$\Rightarrow$ Lösg: $\underline{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = C_1 \cos(\omega_1 t - \delta_1)\begin{pmatrix} 1 \\ +\sqrt{2} \\ 1 \end{pmatrix}$

$+ C_2 \cos(\omega_2 t - \delta_2)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$+ C_3 \cos(\omega_3 t - \delta_3)\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

q in der Basis aus Eigenvektoren

7.2 Übergang zum Kontinuum

Motto: 1, 2, 3, viele, d.h. jetzt N Schwinger

Bewegungsgleichg.: $m\ddot{q}_j + D\left(2q_j - q_{j+1} - q_{j-1}\right) = 0 \quad , j = 1, \dots, N$

$q_0(t) = 0 \ , \ q_{N+1}(t) = 0$ eingespannt am Rand

Ansatz: $q_j(t) = C q_j \cos(\omega t - \delta) = C \sin(\alpha j) \cos(\omega t - \delta)$

mit $\sin((N+1)\alpha) = 0 \Leftrightarrow (N+1)\alpha = \pi k$

$\hookrightarrow \alpha = \frac{\pi k}{N+1} \quad k = 1, 2, \dots$

$$\Rightarrow \Big[-m\omega^2 + \underbrace{2D\left(1 - \cos\left(\frac{\pi k}{N+1}\right)\right)\Big] C \sin(\alpha j)} = 0$$

wegen $D\left(2q_j - q_{j+1} - q_{j-1}\right) = C\cos(\omega t - \delta) D\Big[\sin(\alpha j) - \sin(\alpha(j+1)) + \sin(\alpha j) - \sin(\alpha(j-1))\Big]$

(Trick: Komplexe Darstellung von $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$)

$= 2D\cos(\omega t - \delta)\left(1 - \cos\alpha\right) \sin(\alpha j)$

nicht triviale Lösungen für $C \neq 0$, $\sin(\alpha j) \neq 0$:

$$\omega^2 = \frac{2D}{m}\left(1 - \cos\frac{\pi k}{N+1}\right) \underset{\uparrow}{=} \frac{4D}{m}\sin^2\frac{\pi k}{2(N+1)}$$

$2\sin^2 x = (1 - \cos(2x))$

$$\Rightarrow \omega_k = 2\sqrt{\frac{D}{m}} \sin\frac{\pi k}{2(N+1)} \quad , k = 1, \dots, N$$

$N=1$: $\omega_1 = 2\sqrt{\frac{D}{m}}\sin\frac{\pi k}{2 \cdot 2} = 2\sqrt{\frac{D}{m}}\sin\frac{\pi}{4} = 2\sqrt{\frac{D}{m}}\frac{1}{2}\sqrt{2} = \sqrt{2}\,\omega_0$

$N=2$: $\omega_2 = 2\sqrt{\frac{D}{m}}\sin\frac{\pi k}{2 \cdot 3} \Rightarrow k=1$: $\omega_{2,1} = 2\sqrt{\frac{D}{m}}\sin\frac{\pi}{6} = \sqrt{\frac{D}{m}} = \omega_0$

$k=2$: $\omega_{2,2} = 2\sqrt{\frac{D}{m}}\sin\frac{\pi}{3} = 2\sqrt{\frac{D}{m}}\frac{1}{2}\sqrt{3} = \sqrt{\frac{3D}{m}}$

$N=3$: $\omega_3 = 2\sqrt{\frac{D}{m}}\sin\frac{\pi k}{8} \Rightarrow k=1$: $\omega_{3,3} = 2\sqrt{\frac{D}{m}}\sin\frac{\pi}{8} = 2\sqrt{\frac{D}{m}}\frac{1}{2}\left(2 - \sqrt{2}\right)^{\frac{1}{2}}$

$k=2$: $\omega_{3,2} = 2\sqrt{\frac{D}{m}}\sin\frac{\pi}{4} = 2\sqrt{\frac{D}{m}}\frac{1}{2}\sqrt{2} = \sqrt{\frac{2D}{m}} = \sqrt{2}\,\omega_0$

$k=3$: $\omega_{3,3} = 2\sqrt{\frac{D}{m}}\sin\frac{\pi 3}{8} = 2\sqrt{\frac{D}{m}}\frac{1}{2}\left(1 + \sqrt{2}\right)^{\frac{1}{2}}$

$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2 - \sqrt{2}}$

Nebenrechng

$$\sin(\alpha j) = \frac{1}{2i}\left(e^{i\alpha j} - e^{-i\alpha j}\right)$$

$$\Rightarrow 2\sin(\alpha j) - \sin(\alpha(j+1)) - \sin(\alpha(j-1))$$

$$= \frac{1}{2i}\left(2e^{i\alpha j} - 2e^{-i\alpha j} - e^{i\alpha j}e^{i\alpha} + e^{-i\alpha j}e^{-i\alpha} - e^{i\alpha j}e^{-i\alpha} + e^{-i\alpha j}e^{i\alpha}\right)$$

$$= \frac{1}{2i}\left[e^{i\alpha j}\left(2 - e^{i\alpha} - e^{-i\alpha}\right) - e^{-i\alpha j}\left(2 - e^{-i\alpha} - e^{i\alpha}\right)\right]$$

$$= \frac{1}{2i}\left(2 - e^{i\alpha} - e^{-i\alpha}\right)\left(e^{i\alpha j} - e^{-i\alpha j}\right)$$

$$= \left[2 - \underbrace{\left(e^{i\alpha} + e^{-i\alpha}\right)}_{2\cos\alpha}\right]\sin(\alpha j)$$

$$= 2\left(1 - \cos\alpha\right)\sin(\alpha j)$$

$$\Rightarrow \quad a_j = a_{jk} = \sin\left(\frac{\pi k}{N+1} j\right) \qquad j, k = 1, \dots N$$

$$\Rightarrow \quad q_j(t) = \sum_{k=1}^{N} C_k \sin\left(\frac{\pi k}{N+1} j\right) \cos(\omega_k t - \delta_k)$$

$$= \sum_{k=1}^{N} \sin\left(\frac{\pi k}{N+1} j\right) \left(d_k \cos(\omega_k t) + e_k \sin(\omega_k t)\right)$$

Integrationskonstanten

$\left| \cdot \sin\left(\frac{\pi s}{N+1} j\right) \right.$, $\left| \sum_{j=1}^{N} \right.$

$$d_s = \frac{2}{N+1} \sum_{j=1}^{N} q_j(0) \sin\left(\frac{\pi s}{N+1} j\right), \quad e_s = \frac{2}{N+1} \frac{1}{\omega_s} \sum_{j=1}^{N} \dot{q}_j(0) \sin\left(\frac{\pi s}{N+1} j\right)$$

↳ Fourier-Zerlegung

Übergang zum Kontinuum:

$N \to \infty$ und $l_0 \to 0$: $(N+1) l_0 = l = \text{const}$ (Gesamtlänge)

$m \to 0$ und $l_0 \to 0$: $\frac{m}{l_0} = \rho = \text{const}$ (Massendichte)

$D \to \infty$ und $l_0 \to 0$: $D l_0 = \text{const.}$ (Potenzielle Energie)

Intuition: 

7.3 Wellengleichung

⇒ Bewegungsgleichung:

$$\frac{m}{l_0} \frac{1}{D l_0} \ddot{q}_j = \frac{1}{l_0} \left[\frac{q_{j+1} - q_j}{l_0} - \frac{q_j - q_{j-1}}{l_0} \right]$$

Nun: Limes $l_0 \to 0$: $q_j(t) \to q(x,t)$ mit $x = j l_0$

$$\Rightarrow \frac{m}{l_0} \frac{1}{D l_0} \frac{\partial^2}{\partial t^2} q(x,t) = \lim_{l_0 \to 0} \frac{1}{l_0} \left[\frac{q(x+l_0,t) - q(x,t)}{l_0} - \frac{q(x,t) - q(x-l_0,t)}{l_0} \right]$$

$$= \lim_{l_0 \to 0} \frac{1}{l_0} \left(\left.\frac{\partial q}{\partial x}\right|_{x+\frac{l_0}{2}} - \left.\frac{\partial q}{\partial x}\right|_{x-\frac{l_0}{2}} \right)$$

Wellengleichung

$$= \frac{\partial^2 q}{\partial x^2} \Rightarrow \frac{1}{c^2} \frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\partial^2 q(x,t)}{\partial x^2} \text{ mit } \frac{1}{c^2} = \frac{m}{l_0} \frac{1}{D l_0}$$