

7. Continuum mechanics

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7.1 3 spring coupled oscillators



$$L = T - V$$

$$= \frac{m}{2} \sum_{i=0}^{N+1} \dot{q}_i^2 - \frac{D}{2} \sum_{i=0}^{N+1} (q_i - q_{i-1})^2$$

equation of motion: $m \ddot{q}_j = -D(q_{j-1} - 2q_j + q_{j+1}) = 0 \quad j=1,2,3$

boundary conditions: $q_0(t) = 0 = q_{N+1}(t)$

solution / ansatz: $q_j(t) = C q_j \cos(\omega t - \delta)$

$$\begin{pmatrix} 2\frac{D}{m} & -\frac{D}{m} & 0 \\ -\frac{D}{m} & 2\frac{D}{m} & -\frac{D}{m} \\ 0 & -\frac{D}{m} & 2\frac{D}{m} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \omega^2 \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \text{eigenvalue equation}$$

$$\Rightarrow \lambda_1 = \omega_1^2 = 2\frac{D}{m} \Rightarrow \omega_1 = \sqrt{2} \omega_0, \quad \omega_0 = \sqrt{\frac{D}{m}}, \quad \underline{a}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_{2,3} = \omega_{2,3}^2 = (2 \pm \sqrt{2}) \frac{D}{m} \Rightarrow \omega_2 = \sqrt{2 - \sqrt{2}} \omega_0, \quad \underline{a}^{(2)} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

$$\omega_3 = \sqrt{2 + \sqrt{2}} \omega_0, \quad \underline{a}^{(3)} = \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

7.2 limit to continuum

$$q_j(t) = C \sin(\alpha j) \cos(\omega t - \delta), \quad j=1, \dots, N, \quad q_0(t) = 0 = q_{N+1}(t)$$

with $\alpha = \frac{\pi k}{N+1}$

$$\left[(-m\omega^2 + 2D) \left(1 - \cos\left(\frac{\pi k}{N+1}\right) \right) \right] C \sin(\alpha j) = 0$$

$$\omega_k = 2\sqrt{\frac{D}{m}} \sin\left(\frac{\pi k}{2(N+1)}\right) \quad \text{checked for } k=1,2,3$$

$$\Rightarrow q_j(t) = \sum_{k=1}^N \sin\left(\frac{\pi k}{N+1} j\right) \underbrace{C_k \cos(\omega_k t - \delta_k)}_{[d_k \cos(\omega_k t) + e_k \sin(\omega_k t)]}$$

on sine curves!

limits:

$$N \rightarrow \infty, l_0 \rightarrow 0 : (N+1)l_0 = l = \text{const} \quad (\text{total length})$$

$$m \rightarrow 0, l_0 \rightarrow 0 : \frac{m}{l_0} = \rho = \text{const} \quad (\text{density of mass})$$

$$D \rightarrow \infty, l_0 \rightarrow 0 : D l_0 = \text{const} \quad \begin{array}{l} \text{(total potential energy)} \\ \text{(modulus of elasticity)} \end{array}$$

$$\text{with spatial coordinate: } x = j l_0 \rightarrow q_j(t) \rightarrow \varphi(x, t)$$

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi(x, t) = \frac{\partial^2}{\partial x^2} \varphi(x, t) \quad \text{Wave equation}$$

$$\frac{1}{c^2} = \frac{m}{l_0} \frac{1}{D l_0}$$

$$\Rightarrow a_j = a_{j,k} = \sin\left(\frac{\pi k}{N+1} j\right) \quad j, k = 1, \dots, N$$

$$\Rightarrow q_j(t) = \sum_{k=1}^N C_k \sin\left(\frac{\pi k}{N+1} j\right) \cos(\omega_k t - \delta_k)$$

$$= \sum_{k=1}^N \sin\left(\frac{\pi k}{N+1} j\right) \left(d_k \cos(\omega_k t) + e_k \sin(\omega_k t) \right)$$

Integrationskonstanten

$$\left(\sin\left(\frac{\pi k}{N+1} j\right) \right) \sum_{j=1}^N$$

$$d_k = \frac{2}{N+1} \sum_{j=1}^N q_j(0) \sin\left(\frac{\pi k}{N+1} j\right), \quad e_k = \frac{2}{N+1} \frac{1}{\omega_k} \sum_{j=1}^N \dot{q}_j(0) \sin\left(\frac{\pi k}{N+1} j\right)$$

↳ Fourier-Zerlegung

Übergang zum Kontinuum:

$$N \rightarrow \infty \text{ und } l_0 \rightarrow 0 : (N+1)l_0 = l = \text{const (Gesamtlänge)}$$

$$m \rightarrow 0 \text{ und } l_0 \rightarrow 0 : \frac{m}{l_0} = \rho = \text{const (Massendichte)}$$

$$D \rightarrow \infty \text{ und } l_0 \rightarrow 0 : D l_0 = \text{const.} \quad \begin{matrix} = \gamma = D \\ \text{(Potenzielle Energie)} \\ \text{(Elastizitätsmodul)} \end{matrix}$$

Interferenz:

mm □ mm

→

mm □ mm

→

7.3 Wellengleichung

⇒ Bewegungsgleichung:

$$\frac{m}{l_0} \frac{1}{D l_0} \ddot{\varphi}_j = \frac{1}{l_0} \left[\frac{\varphi_{j+1} - \varphi_j}{l_0} - \frac{\varphi_j - \varphi_{j-1}}{l_0} \right]$$

Nimm: Lim $l_0 \rightarrow 0$: $\varphi_j(t) \rightarrow \varphi(x, t)$ mit $x = j l_0$

$$\Rightarrow \frac{m}{l_0} \frac{1}{D l_0} \frac{\partial^2}{\partial x^2} \varphi(x, t) = \lim_{l_0 \rightarrow 0} \frac{1}{l_0} \left[\frac{\varphi(x+l_0, t) - \varphi(x, t)}{l_0} - \frac{\varphi(x, t) - \varphi(x-l_0, t)}{l_0} \right]$$

$$= \lim_{l_0 \rightarrow 0} \frac{1}{l_0} \left(\frac{\partial \varphi}{\partial x} \Big|_{x+l_0} - \frac{\partial \varphi}{\partial x} \Big|_{x-l_0} \right)$$

$$= \frac{\partial^2 \varphi}{\partial x^2} \Rightarrow \frac{1}{c^2} \frac{\partial^2 \varphi(x, t)}{\partial t^2} = \frac{\partial^2 \varphi(x, t)}{\partial x^2} \text{ mit } \frac{1}{c^2} = \frac{m}{l_0 D}$$

Wellengleichung

mit $\frac{m}{l_0} \frac{1}{Dl_0} = \frac{g}{Dl_0} =: \frac{1}{c^2}$:

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$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(x,t) = \frac{\partial^2}{\partial x^2} \psi(x,t) \quad \text{Wellengleichung}$$

$$[c^2] = \left[\frac{Dl_0}{g} \right] = \left[\frac{\frac{F}{l_0} l_0}{g} \right] = \left[\frac{F l_0}{g} \right] = \frac{\frac{kg}{s^2}}{\frac{kg}{m}} = \frac{m^2}{s^2} \Rightarrow [c] = \frac{m}{s}$$

c : Phasengeschwindigkeit

Lösungen:

(i) D'Alembertsche Lösung: $\psi(x,t) = f(x \pm ct)$

im mit bewegtem Koordinatensystem: stehende Welle

(ii) Bernoullische Lösung: $\psi(x,t) = g(x) h(t)$

$$\Rightarrow \frac{1}{c^2} g(x) \ddot{h}(t) = g''(x) h(t) \Rightarrow \frac{1}{c^2} \frac{\ddot{h}(t)}{h(t)} = \frac{g''(x)}{g(x)} = -k^2 = \text{const.}$$

Also: $\ddot{h}(t) = -c^2 k^2 h(t)$ und $g''(x) = -k^2 g(x)$

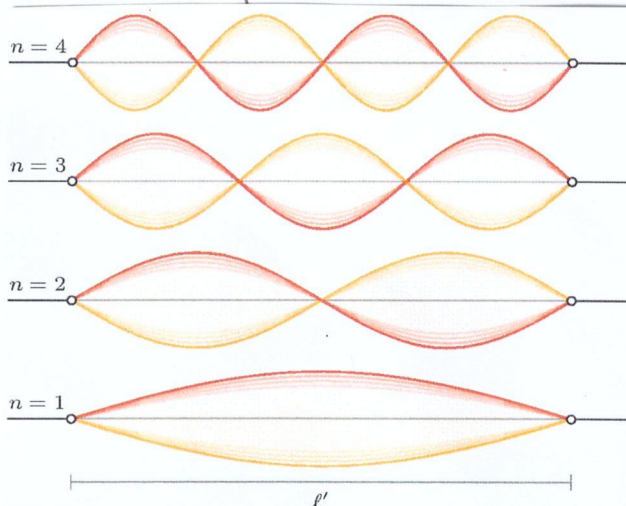
$\Rightarrow h(t) = h_1 \cos(kt) + h_2 \sin(kt)$ und $g(x) = g_1 \cos(kx) + g_2 \sin(kx)$

Randbedingung des festgehaltenen, schwingenden Mediums (unendliche Kette):

$g(0) = g(l) = 0 \Rightarrow g_1 = 0, \sin(kl) = 0 \Rightarrow kl = n\pi \quad n=1,2,\dots$

mit $k_n c t = \omega_n t$: **Eigenfrequenzen** $k_n = \frac{n\pi}{l}$ $\omega_n = n \frac{\pi c}{l}$

$\Rightarrow \psi_n(x,t) = \sin \frac{n\pi x}{l} \left(d_n \cos(\omega_n t) + e_n \sin(\omega_n t) \right)$ **Eigene Schwingung**



4. Harmonische

3. Harmonische

2. Harmonische

Grundschwingung

⇒ allgemeine Lösung:

$$\varphi(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(d_n \cos(\omega_n t) + e_n \sin(\omega_n t) \right)$$

$$d_n = \frac{2}{l} \int_0^l \varphi(x, 0) \sin \frac{n\pi x}{l} dx \quad e_n = \frac{2}{l\omega_n} \int_0^l \dot{\varphi}(x, 0) \sin \frac{n\pi x}{l} dx$$

Bsp.: Membran Schwingungen

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \quad \text{mit } \varphi = \varphi(x, y, t)$$

Randbedingung einer rechteckigen Membran



$$x=0, x=a, y=0, y=b$$

$$\varphi(0, y, t) = \varphi(a, y, t) = \varphi(x, 0, t) = \varphi(x, b, t) = 0 \quad \forall t \quad (\text{fest eingespannt})$$

$$\text{Produktansatz: } \varphi(x, y, t) = \sin(kx) \sin(\hat{k}y) \left[d \cos(\omega t) + e \sin(\omega t) \right]$$

$$\text{mit } k = \frac{n\pi}{a}, \quad \hat{k} = \frac{s\pi}{b}, \quad n, s = 1, 2, \dots$$

$$\hookrightarrow \text{Eigenfrequenzen: } \omega_{n,s} = \pi c \sqrt{\frac{n^2}{a^2} + \frac{s^2}{b^2}} \quad \text{irrationale Vielfache von } \omega_{1,1} = \pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

allgemeine Lösung:

$$\varphi(x, y, t) = \sum_{n,s=1}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{s\pi y}{b} \left[d_{n,s} \cos(\omega_{n,s} t) + e_{n,s} \sin(\omega_{n,s} t) \right]$$

mit Fourier Koeffizienten:

$$d_{n,s} = \frac{4}{ab} \int_0^a \int_0^b \varphi(x, y, 0) \sin \frac{n\pi x}{a} \sin \frac{s\pi y}{b} dx dy$$

$$e_{n,s} = \frac{4}{ab \omega_{n,s}} \int_0^a \int_0^b \dot{\varphi}(x, y, 0) \sin \frac{n\pi x}{a} \sin \frac{s\pi y}{b} dx dy$$

Frequenzen (i) i.A. irrationale Vielfache ⇒ kein harmonisches Obertöne

$$\text{etwa: für } a=b: \omega_{1,2} = \pi c \sqrt{\frac{5}{a^2}} = \sqrt{5} \omega_{1,1}$$

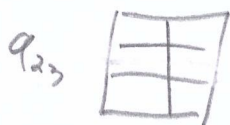
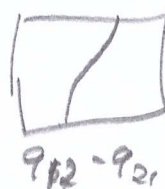
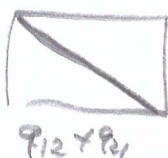
(ii) Entartung: Schwingungen mit gleicher Eigenfrequenz $\omega_{n_1,s_1} = \omega_{n_2,s_2}$:

$$\frac{n_1^2}{a^2} + \frac{s_1^2}{b^2} = \frac{n_2^2}{a^2} + \frac{s_2^2}{b^2} \Leftrightarrow \frac{a^2}{b^2} = \frac{n_1^2 - n_2^2}{s_2^2 - s_1^2} \Rightarrow \text{Bed. } \frac{a^2}{b^2} \in \mathbb{Q}$$

Knotenlinien: $\psi(x, y, t) = 0$

(a) parallel zum Rand: $x = 0, \frac{a}{n}, \frac{2a}{n}, \dots, a$

$$y = 0, \frac{b}{s}, \frac{2b}{s}, \dots, b$$



(b) $a = b$, $d_{12}/d_{21} = q_{12}/q_{21} = u$, $\omega_{12} = \omega_{21} = \bar{\omega}$, $\bar{\omega}$ Frequenz von die Eigenleistung q_{12}, q_{21}

$$\psi(x, y, t) = \left[u \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} + \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right]$$

$$(d_{21} \cos(\bar{\omega} t) + e_{21} \sin(\bar{\omega} t))$$

$$\Rightarrow [\dots] = 0 \Leftrightarrow u \cos \frac{\pi y}{a} + \cos \frac{\pi y}{a} = 0$$

$$\uparrow$$

$$\sin(2\pi) = 2 \sin \pi \cos \pi$$

$$\left(u \sin \frac{\pi x}{a} \cdot 2 \sin \frac{\pi y}{a} \cos \frac{\pi y}{a} + \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \sin \frac{\pi y}{a} \right)$$

$$= 2 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \left(u \cos \frac{\pi y}{a} + \cos \frac{\pi y}{a} \right)$$

$u = \pm 1$: Diagonale

(c) kreisförmige Membran: $\rightarrow$ Polarkoordinaten (r, φ)

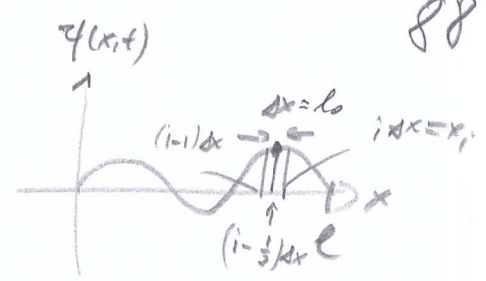
Besselsche Gleichung: $\psi(r, \varphi) = R(r) \phi(\varphi)$

$$\hookrightarrow \phi'' + \lambda^2 \phi = 0 \quad \text{Winkelanteil}$$

$$r^2 R'' + r R' + (k^2 r^2 - \lambda^2) R = 0 \quad \text{Radialanteil (Bessel-DGL)}$$

Linear der Lagrange-Funktion:

$$T = \sum_{i=1}^N \frac{m_i}{2} \left(\frac{dq_i}{dt} \right)^2 = \sum_{i=1}^N \frac{\rho \Delta x}{2} \dot{q}_i^2$$



m_i = Masse bei $x_i = (i - \frac{1}{2}) \Delta x$ $i = 1, \dots, N$, ρ : Massendichte $\frac{m}{\Delta x} = \frac{m}{l_0}$

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{\rho \Delta x}{2} \dot{q}_i^2 = \frac{\rho}{2} \lim_{N \rightarrow \infty} \sum_{i=1}^N \Delta x \left(\frac{\partial y(x_i, t)}{\partial t} \right)^2 = \frac{\rho}{2} \int_0^l dx \left(\frac{\partial y(x, t)}{\partial t} \right)^2$$

Für V : $\lim_{\Delta x \rightarrow 0} \frac{q_{i+1}(t) - q_i(t)}{\Delta x} = \frac{\partial y(x, t)}{\partial x}$

$$\Rightarrow V = \lim_{N \rightarrow \infty} \frac{D}{2} \sum_{i=1}^{N-1} (q_{i+1} - q_i)^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^{N-1} \frac{D(\Delta x)^2}{2} \left(\frac{y(x_{i+1}, t) - y(x_i, t)}{(\Delta x)^2} \right)^2$$

geometrisch: $\Delta s = \sqrt{(\Delta x)^2 + (y(x_{i+1}, t) - y(x_i, t))^2}$
 $\uparrow$
 Abstand $i \leftrightarrow i+1$ $\approx \Delta x \left[1 + \frac{(y(x_{i+1}, t) - y(x_i, t))^2}{2(\Delta x)^2} \right]$

Kraft proportional zu $\Delta s - \Delta x$

Verspannung $\Rightarrow V = u - 2u(\Delta s - \Delta x)^2$

$\downarrow$
 $P = D \Delta x = \text{const}$

$\downarrow$
 $\frac{1}{2} D \Delta x \int_0^l dx \left(\frac{\partial y(x, t)}{\partial x} \right)^2$
 $\frac{1}{2} \sum_{i=1}^N l_0 \rightarrow \int_0^l dx$

$P = D \Delta x = D l_0$: Verspannung
 elastische Feder

$$\Rightarrow L = T - V = \int_0^l dx \left[\frac{\rho}{2} \left(\frac{\partial y(x, t)}{\partial t} \right)^2 - \frac{D}{2} \left(\frac{\partial y(x, t)}{\partial x} \right)^2 \right]$$

$L(y, y')$ Lagrange-Dichte

• Hamiltonsches Prinzip:

$$\delta S = \delta \int_{t_1}^{t_2} dt \int_0^l dx L(y, \dot{y}, y', x, t) = 0$$

$$\delta L = \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} + \frac{\partial L}{\partial y'} \delta y'$$

• Variations: vgl. 3.2 Nach dem Prinzip der kleinsten Wirkung

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$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}'} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Partielle DGL!
(vgl. $\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0, j=1, \dots, N$)

mit: $\mathcal{L} = (g \dot{\varphi}^2 - D \varphi'^2)/2$ folgt:

$$g \frac{\partial^2 \varphi}{\partial t^2} - D \frac{\partial^2 \varphi}{\partial x^2} = 0$$

Wellengleichung (frei Schwingen)

$$\underbrace{\frac{g}{D}}_{\frac{1}{c^2}} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} \quad \left(+ \varphi f(x,t) \right)$$

äußere Kraftdichte

(Energie: $\Delta V = -\varphi f \Delta x$)

etwa: $f = -g \left(\frac{g}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{D}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + f(x,t) \varphi \right)$

Typ: Wellengleichung für Feld φ (Feldgleichung)

$= T - V$
hin. Abweichung

• 2/30: $\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)}_{\Delta} \varphi(x,y,z,t)$

Δ : D'Alembert-Operator