7.3 Wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,t) = \frac{\partial^2}{\partial x^2}\psi(x,t) \qquad c\text{: (phase) velocity}$$

↳ D'Alembert | ansatz solution $\psi(x,t) = f(x \pm ct)$

↳ standing waves in co-moving frame

↳ Bernoulli | solution ansatz: $\psi(x,t) = g(x)\, h(t)$

$$\Rightarrow \ddot{h}(t) = -c^2 k^2 h(t) \quad \text{and} \quad g''(x) = -k^2 g(x)$$

e.g. (i) guitar string:

$$\psi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}x\right)\left(d_n \cos(\omega_n t) + e_n \sin(\omega_n t)\right)$$

(ii) 2D rectangular | membrane drum : $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,y,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,t)$

$$\hookrightarrow \psi(x,y,t) = \sum_{n,s=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{s\pi}{b}y\right)\left[d_{ns}\cos(\omega_{ns}t) + e_{ns}\sin(\omega_{ns}t)\right]$$

$$\omega_{ns} = \pi c\sqrt{\frac{n^2}{a^2} + \frac{s^2}{b^2}}$$

• Limit of Lagrange function:

$$L = T - V = \int_0^l dx \underbrace{\left[\frac{\rho}{2}\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2 - \frac{P}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 + f(x,t)\psi(x,t)\right]}_{\mathcal{L}\ \text{: Lagrange density}}$$

$$\hookrightarrow \delta S = \delta\int_{t_1}^{t_2} dt\, L = 0$$

$$\Rightarrow \frac{\partial}{dt}\frac{\partial\mathcal{L}}{\partial\dot\psi} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\psi'} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \qquad \left(\text{cf. } \frac{d}{dt}\frac{\partial L}{\partial\dot q_j} - \frac{\partial L}{\partial q_j} = 0\right)$$

$f(x,t)$ in case of external force

total force $F(t) = \int_l dx\, f(x,t)$

Beispiel 3: Kreisförmige Membran in Schwingung

Start: Wellengleichung (2D): $u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$

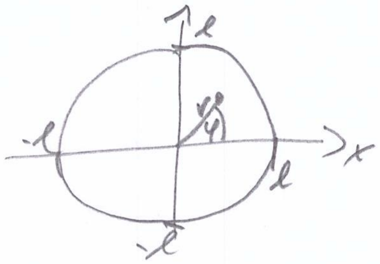
$$\underbrace{\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}}_{\to \Delta u(x,y,t)} = \frac{1}{c^2} \frac{\partial^2 u(x,y,t)}{\partial t^2}$$

Laplace-Op.

mit Randbedingung: $u(r=l, \varphi, t) = 0$ in Polarkoordinaten

$x = r\cos\varphi$

$y = r\sin\varphi$



Anfangsbedingung: $u(r,\varphi,0)$ und $\frac{\partial}{\partial t} u(r,\varphi,0)$

Idee: 2x Separationsansatz + Rekursionsformel

$u(r,\varphi,t) = f(r,\varphi)\, T(t)$

$$\Rightarrow \frac{\Delta f(r,\varphi)}{f(r,\varphi)} = \frac{1}{c^2} \frac{\ddot{T}(t)}{T(t)} = -k^2 = \text{const.}$$

$\Rightarrow T(t) = A\cos(\omega t) + B\sin(\omega t)$, $\omega = kc$

$\Rightarrow$ Ortsanteil: $\Delta f(r,\varphi) + k^2 f(r,\varphi) = 0$ mit $f(l,\varphi) = 0$

in Polarkoordinaten:

$$\Delta f(r,\varphi) = \frac{1}{r}\frac{\partial}{\partial r} f + \frac{\partial^2}{\partial r^2} f + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} \quad \left[= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2}\right]$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2}{\partial r^2} f + \frac{1}{r}\frac{\partial}{\partial r} f + k^2 f = 0$$

Separationsansatz: $f(r,\varphi) = R(r)\,\Phi(\varphi)$

$$\frac{\partial^2 \Phi}{\partial \varphi^2}\frac{R}{r^2} + \Phi\left(\frac{\partial^2}{\partial r^2} R + \frac{1}{r}\frac{\partial R}{\partial r} + k^2 R\right) = 0$$

$$\frac{1}{\phi}\frac{\partial^2\phi}{\partial\varphi^2} = -\frac{1}{R}\left(r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + k^2 r^2 R\right) = -\lambda^2 = \text{const}$$

$$\Rightarrow \underbrace{\frac{\partial^2\phi}{\partial\varphi^2} + \lambda^2\phi = 0}_{\text{Winkelanteil}}, \quad \underbrace{r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (k^2 r^2 - \lambda^2)R = 0}_{\text{Radialanteil}}$$

$$\hookrightarrow \phi(\varphi) = C\cos(\lambda\varphi + \varphi_0) \overset{!}{=} \phi(\varphi + 2\pi) \quad \text{Periodizität in } \varphi$$

$$\Rightarrow \cos(\lambda(\varphi + 2\pi) + \varphi_0) \overset{!}{=} \cos(\lambda\varphi + \varphi_0 + 2\pi\lambda)$$

$$\Rightarrow \lambda = m \in \mathbb{N}.$$

Bleibt noch: $r^2 R'' + r R' + (k^2 r^2 - \lambda^2) R = 0$

mit $x := kr \Rightarrow R' = k\frac{dR}{dx}, \quad R'' = k^2\frac{d^2R}{dx^2}$:

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - \lambda^2)R = 0 \qquad \text{Bessel'sche DGL}$$

Lösungen: $R(x) = x^\rho \sum_{s=0}^{\infty} \alpha_s x^s$ (vgl. TP2 Kap. 7.3(a))

$$\Rightarrow \frac{dR}{dx} = \sum_{s=0}^{\infty} \alpha_s (\rho+s)\, x^{(\rho+s)-1}$$

$$\frac{d^2R}{dx^2} = \sum_{s=0}^{\infty} \alpha_s (\rho+s)(\rho+s-1)\, x^{(\rho+s)-2}$$

Einsetzen:

$$\sum_{s=0}^{\infty} \alpha_s \big[\underbrace{(\rho+s)(\rho+s-1) + (\rho+s)}_{(\rho+s)^2} - \lambda^2\big] x^{\rho+s} + \sum_{s=0}^{\infty} \alpha_s x^{\rho+s+2} = 0$$

$$\Rightarrow \alpha_0[\rho^2 - \lambda^2]x^\rho + \alpha_1[(\rho+1)^2 - \lambda^2]x^{\rho+1} + \sum_{s=2}^{\infty}\left\{\alpha_s[(\rho+s)^2 - \lambda^2] + \alpha_{s-2}\right\} x^{\rho+s} = 0$$

Koeffizienten vergleichen:

x^ρ: $\alpha_0[\rho^2 - \lambda^2] = 0 \qquad \Rightarrow \rho = \pm\lambda \quad (\alpha_0 \neq 0)$

$x^{\rho+1}$: $\alpha_1[(\rho+1)^2 - \lambda^2] = 0 \qquad \Rightarrow \alpha_1[(\pm\lambda+1)^2 - \lambda^2] = 0 \Rightarrow \alpha_1 = 0$

$x^{\rho+s}$: $\alpha_s[(\rho+s)^2 - \lambda^2] + \alpha_{s-2} = 0 \qquad (s \geq 2)$

$$\Rightarrow \alpha_s = \frac{\alpha_{s-2}}{\lambda^2 - (\pm\lambda+s)^2} = \frac{\alpha_{s-2}}{s(\mp 2\lambda - s)} \qquad \text{Rekursionsformel}$$

wegen $\alpha_1 = 0$ und Rekursionsformel: alle ungeraden $\alpha_i = \alpha_{2n+1} = 0$ §2

$s = +\lambda$: $\alpha_2 = -\dfrac{\alpha_0}{2(2\lambda+2)}$, $\alpha_4 = -\dfrac{\alpha_2}{4(2\lambda+4)} = \dfrac{\alpha_0}{2\cdot 4(2\lambda+2)(2\lambda+4)}$

$$\alpha_6 = -\frac{\alpha_0}{2\cdot 4\cdot 6\,(2\lambda+2)(2\lambda+4)(2\lambda+6)} \quad \dots$$

mit Festlegung: $\alpha_0 := \dfrac{1}{2^\lambda \lambda!}$ setze ($\lambda = m$)

$$J_m(x) = x^m \sum_{s=0}^{\infty} \alpha_s \underset{\underset{s=2r}{\uparrow}}{x^s} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(m+r)!}\left(\frac{x}{2}\right)^{m+2r} \quad \text{Bessel-Funktionen}$$

$s = -\lambda$: $J_{-\lambda}(x) = x^{-\lambda} \sum_{s=0}^{\infty} \alpha_s x^s$ Achtung: $\lambda = m \Rightarrow \alpha_s = \dfrac{\alpha_{s-2}}{s(2m-s)}$ singulär für $s = 2m$

$(\lambda! = \Gamma(\lambda+1))$

$$\Rightarrow R(r) \underset{\underset{x=kr}{\uparrow}}{=} D\, J_m(kr),$$

$$\Rightarrow u(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} u_{nm}$$

$$= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left[A_{nm}\cos(\omega_{nm}t) + B_{nm}\sin(\omega_{nm}t)\right] \cos(m\varphi + \varphi_0)\, J_m(k_{mn} r)$$

Eigenschaften:

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$$

$$\frac{dJ_m(x)}{dx} = -\frac{m}{x} J_m(x) + J_{m-1}(x)$$

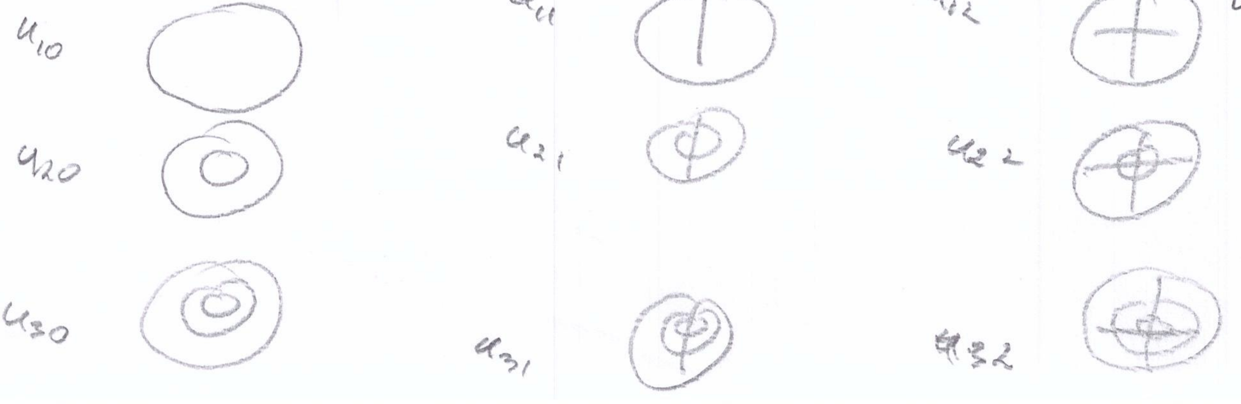
Nullstellen: $J_m(\mu_{mn}) = 0$

$$\Rightarrow kl = \mu_{mn} = \frac{\omega}{c} l$$

$$\Rightarrow \omega_{mn} = \frac{c}{l} \mu_{mn}$$

kein rationales Verhältnis

Knotenlinien:



Erweiterung der Lagrange-Dichte auf 3 Raumdimensionen: $x \to \underline{x}$

$q \to \underline{q} = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}$

$$L = \int_V d^3r \left[\frac{\rho}{2} \dot{\underline{q}}^2(\underline{x},t) - V(\underline{q}, \underline{\nabla} \circ \underline{q}) \right]$$

mit potenzieller Energiedichte $V(\underline{q}, \underline{\nabla}\circ\underline{q}) = V^{\text{außen}}(\underline{q},\underline{x}) + V^{\text{innen}}(\underline{\nabla}\circ\underline{q})$

$$V^{\text{außen}}(\underline{q},\underline{x}) = -\underline{f}(\underline{x},t)\cdot\underline{q}(\underline{x},t)$$

$$\underbrace{\underline{\nabla}\circ\underline{q}}_{\text{dyadisches Produkt}} = \underline{\underline{\eta}} = \begin{pmatrix} \partial_x q_x & \partial_x q_y & \partial_x q_z \\ \partial_y q_x & \partial_y q_y & \partial_y q_z \\ \partial_z q_x & \partial_z q_y & \partial_z q_z \end{pmatrix}$$

$$\Rightarrow V^{\text{innen}} \approx V^{\text{innen}}(0) + \sum_{ij=1}^{3} \frac{\partial V^{\text{innen}}}{\partial \eta_{ij}} \eta_{ij} + \frac{1}{2} \sum_{\substack{ij=1 \\ kl}}^{3} \underbrace{\frac{\partial^2 V^{\text{innen}}}{\partial \eta_{ij} \partial \eta_{kl}}}_{=: C_{ijkl}} \eta_{ij} \eta_{kl}$$

Elastizitätstensor