0. Overview

- analytical mechanics: derive equations of motion using different frameworks of varying mathematical complexity
- deterministic systems: – determined by initial conditions
  – causal : generated by forces

1. Newtonian mechanics

1.1 Approach

Kinematics & dynamics of systems of mass point without constraints

Kinematics: framework using energy instead of forces "geometry of motion"

dynamics: motion based on forces

1.2 Newton's laws of motion

I: A body remains at rest or in motion at a constant speed in a straight line, unless it is acted upon by a force

II: The net force on a body is equal (i) to the bodies acceleration multiplied by its mass (ii) to rate of change of its momentum.

III: If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

IV: Forces obey the superposition principle.

- event: $(\underline{r}(t), t) \in \mathbb{R}^3 \times \mathbb{R}$
- dynamic variable: $\underline{r}(t)$: trajectory

$$\underline{v}(t) = \frac{d}{dt}\underline{r}(t) = \dot{\underline{r}}(t) \quad \text{velocity}$$

- I $\Leftrightarrow$ system of inertia (Galilean transformation)

$$\underline{r}(t) = \underline{\underline{R}}\,\underline{r}'(t') + \underline{v}_0 t + \underline{s}_0$$

$$t = t' + t_0 \quad , \underline{\underline{R}} : \text{rotational matrix}$$

- Newton's laws of motion are Galilean invariant
  ↳ existence of absolute space
  ↳ all inertial frames share a universal time

- II $\Leftrightarrow$ $\underline{F} = m \frac{d^2}{dt^2} \underline{r}(t)$ has a unique solution

  for initial conditions $(\underline{v}_0, t_0) \Rightarrow \underline{r}(t; \underline{r}_0, t_0)$

- III $\Leftrightarrow$ no external forces: overall momentum is conserved

- IV $\Leftrightarrow$ force add like vectors

- mass : a measure of the body's inertia = gravitational mass

$$\underline{F}^{(12)} = -G\, m^{(1)} m^{(2)} \frac{\underline{r}^{(1)} - \underline{r}^{(2)}}{|\underline{r}^{(1)} - \underline{r}^{(2)}|^3}, \quad G = 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$$

2 D'Alembert's principle

2.1 Constraints and forces

System of N mass points $\rightarrow$ $3N$ degrees of freedom (without constraints)

idea: reduction of degrees of freedom by constraints.

(i) holonomic: $f_\mu(\underline{r}_1, \dots, \underline{r}_N, t) = 0 \quad , \mu = 1, \dots, \Lambda \Rightarrow f = 3N - \Lambda$

total differential: $df_\mu = \sum_{i=1}^{N} \underline{\nabla}_{r_i} f_\mu \cdot d\underline{r}_i + \frac{\partial f_\mu}{\partial t} dt = 0$

(ii) nonholonomic: $\sum_{i=1}^{N} \underline{a}_{\mu i}(\underline{r}_1, \dots, \underline{r}_N, t) \cdot d\underline{r}_i + \underline{a}_{\mu 0}(\underline{r}_1, \dots, \underline{r}_N, t)\, dt = 0$

but cannot be written as $df_\mu = \dots = 0$ (even with an integrating factor: $\nexists g_\mu : \underline{a}_{\mu i} = g_\mu \underline{\nabla}_{r_i} f_\mu$

(meaning: motion locally restricted)

(a) rheonomic: explicitly time dependent

(b) scleronomic: not explicitly time dependent

equations of motion: $m_i \ddot{\underline{r}}_i = \underbrace{\underline{F}_i}_{\text{external}} + \underbrace{\sum_{j=1}^{N} \underline{F}_{ij}}_{\text{internal forces}} =: \underline{X}_i \;, \; i = 1, \dots, N$

$\hookrightarrow$ solution given constraints.

Idea: turn constraints into forces governing the motion:

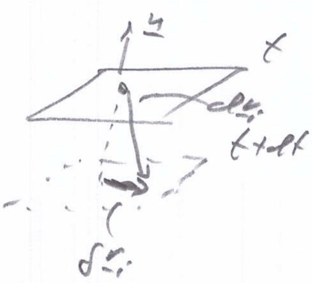
$$m_i \ddot{\underline{r}}_i = \underline{X}_i + \underline{Z}_i$$

2.2 Virtual displacements

- (hypothetical) deviation from trajectory obeying the constraints (for fixed time)
- ex. motion on oscillating plane: $f = \underline{n} \cdot (\underline{r} - \underbrace{\underline{r}_0(t)}_{\text{motion of plane}}) = 0$

$\Rightarrow df = \underline{n} \cdot (d\underline{r}_i - \underline{v}_0(t)\, dt) = 0 \quad$ with $\underline{v}_0 = \frac{d}{dt}\underline{r}_0(t)$

$\Rightarrow \underline{n} \cdot d\underline{r}_i = \underline{v}_0(t)\, dt \quad$ (motion of fixed not zero)



but $\delta f = \underline{n} \cdot \delta \underline{r}_i \underset{\substack{\uparrow \\ \text{fixed time}}}{=} 0 \quad \Rightarrow \quad \delta \underline{r}_i \perp \underline{n}$

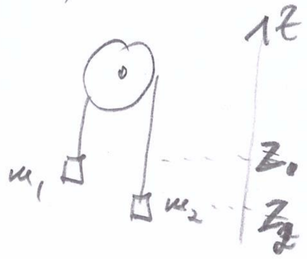
2.3 D'Alembert's principle

virtual work: $\sum_{i=1}^{N} \underline{Z}_i \cdot \delta \underline{r}_i = 0 \Rightarrow$ D'Alembert's principle: $\sum_{i=1}^{N} (m_i \ddot{\underline{r}}_i - \underline{X}_i) \cdot \delta \underline{r}_i = 0$

(notes no constraints explicitly included)

ex: (i) $\underline{Z}_i \cdot \delta \underline{r}_i = \lambda \underbrace{\underline{\nabla}_r f}_{=\underline{n}} \cdot \underbrace{\delta \underline{r}}_{\parallel \text{ to plane}} = 0$

(ii) Atwood's machine: m_1 & m_2 connected via a pulley by a string



$$z_1 + z_2 = \text{const} \Rightarrow \ddot{z}_1 = -\ddot{z}_2, \quad \delta z_1 = -\delta z_2$$

$$\Big(m_1 \ddot{z}_1 - \underbrace{(-m_1 g)}_{X_1}\Big)\delta z_1 + \Big(m_2 \ddot{z}_2 - \underbrace{(-m_2 g)}_{X_2}\Big)\delta z_2 = 0$$

$$\Rightarrow \underbrace{\left[(m_1+m_2)\ddot{z}_1 + (m_1 - m_2) g\right]}_{\stackrel{!}{=}0} \delta z_1 = 0 \text{ for arbitrary } \delta z_1$$

$$\Rightarrow \ddot{z}_1 = -\frac{m_1 - m_2}{m_1 + m_2} g$$

general notation: $\underline{r}_i \to x_j$, $\underline{X}_i \to K_j$, $\underline{a}_{\mu i} \to \phi_j^\mu$, $\underline{\nabla}_i f_\mu \to \phi_j^\mu$

$$i = 1, \dots, N, \quad j = 1, \dots, 3N$$

$$\Rightarrow \sum_{j=1}^{3N} (m_j \ddot{x}_j - K_j)\, \delta x_j = 0 \quad \text{given} \quad \sum_{j=1}^{3N} \phi_j^\mu \delta x_j = 0$$

idea: Lagrange multipliers (add constraints to equation of motion)

$$\hookrightarrow \sum_{j=1}^{3N} \left(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu \phi_j^\mu\right) \delta x_j = 0$$

choose / define $\lambda_1(t), \dots, \lambda_\Lambda(t)$ such that $m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu \phi_j^\mu = 0$, $j = 1, \dots, \Lambda$

insert into remaining equations: $\sum_{j=\Lambda+1}^{3N} \underbrace{\left(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu(t) \phi_j^\mu\right)}_{=0} \underbrace{\delta r_j}_{\text{free}} = 0$

$\Rightarrow$ Lagrange's equation of the first kind:

$$m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_\mu(t) \phi_j^\mu = 0 \qquad j = 1, \dots, 3N$$

2.4 Generalized coordinates

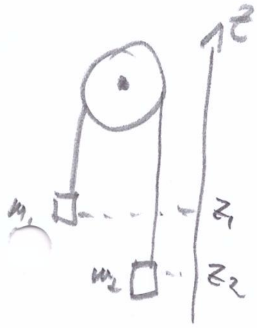
set $\{q_1, \dots, q_f\}$ of independent coordinates that obey given constraints

idea: express virtual displacements $\delta \underline{r}_i \ (i=1,\dots,N)$ by $\delta q_k \ (k=1,\dots,f)$

$\delta x_j \ (j=1,\dots,3N)$

with $f = |\text{degrees of freedom}|$.

case study Atwood's machine



(i) Newton $\underline{F} = m\underline{a}$

(ii) D'Alembert $\sum_{i=1}^{N} \underline{Z}_i \cdot \delta \underline{r}_i = 0 = \sum_{i=1}^{N} (m_i \ddot{\underline{r}}_i - \underline{X}_i) \delta \underline{r}_i$

(iii) Lagrange I $m_j \ddot{x}_j - K_j - \sum_{r=1}^{k} \lambda_r \frac{\partial f_r}{\partial x_j} = 0$

$$\Bigg\} \ \ddot{z}_1 = -\frac{m_1 - m_2}{m_1 + m_2} g$$

2.5 Lagrangian equation of 2nd kind

idea: express $\underline{r}_i$, $\dot{\underline{r}}_i$, $\delta\underline{r}_i$ by generalized coordinates q_j, $\dot{q}_j$, δq_j, $j=1,\dots,f$
$i=1,\dots,N$

e.g.: $\delta\underline{r}_i = \sum_{j=1}^{f} \left(\frac{\partial}{\partial q_j}\underline{r}_i\right)\delta q_j$, $\frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j} = \frac{\partial \underline{r}_i}{\partial q_j}$, $\frac{d}{dt}\left(\frac{\partial}{\partial q_j}\underline{r}_i\right) = \frac{\partial \dot{\underline{r}}_i}{\partial q_j}$

Defining the Lagrangian / Lagrange function: $L = T - V$
with $T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\underline{r}}_i^2$ and $Q_j = -\frac{\partial V}{\partial q_j}$ (generalized forces):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad , j=1,\dots,f$$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

b) $T = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2)$

$$V = \frac{1}{2} m \frac{g}{l} (q_1^2 + q_2^2) + \frac{1}{2} k (q_1 - q_2)^2$$

matrix form: $\underline{\underline{T}} = \{T_{jk}\}_{j,k=1,2} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

$$\underline{\underline{V}} = \{V_{jk}\}_{j,k=1,2} = \begin{pmatrix} m\frac{g}{l} + k & -k \\ -k & m\frac{g}{l} + k \end{pmatrix}$$

equation of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad , j = 1,2$

$$m\ddot{q}_1 + m\frac{g}{l} q_1 + k(q_1 - q_2) = 0$$

$$m\ddot{q}_2 + m\frac{g}{l} q_2 - k(q_1 - q_2) = 0$$

using the ansatz: $q_j = A_j e^{i\omega t}$:

eigenvalue equation: $\underbrace{\begin{pmatrix} m\frac{g}{l} + k & -k \\ -k & m\frac{g}{l} + k \end{pmatrix}}_{\underline{\underline{V}}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \underbrace{m\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}_{\omega^2 \underline{\underline{T}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}$

2.6 Normal modes (continued)

case: 2 coupled pendula

$$m\ddot{q}_1 + m\frac{g}{l}q_1 + k(q_1-q_2)=0$$
$$m\ddot{q}_2 + m\frac{g}{l}q_2 - k(q_1-q_2)=0$$

} derived via Lagrange II

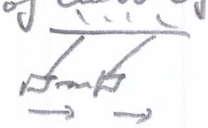
↳ transform into eigenvalue/-vector problem:

Eigenvalues: $\omega_1^2 = \frac{g}{l}$ Eigenvectors: $\begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$ $\begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \end{pmatrix} = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

⇒ Solutions: $Q_1 = \frac{1}{\sqrt{2m}}(q_1+q_2)$: motion of centre of mass

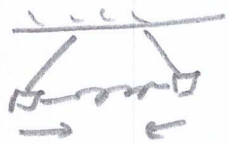
in-phase



normal modes

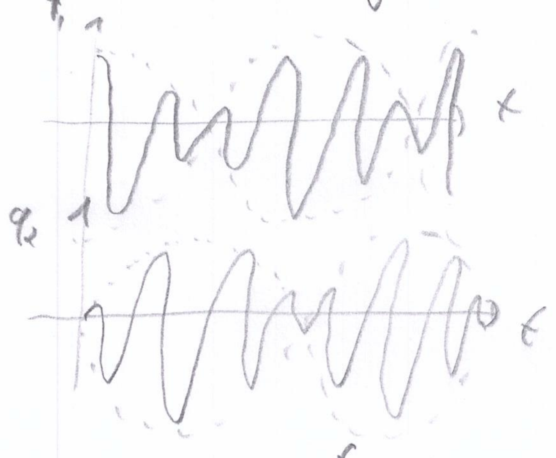
$Q_2 = \frac{1}{\sqrt{2m}}(q_1-q_2)$: relative motion

anti-phase



$\left(q_j(t) = \sum_\alpha A_j^{(\alpha)} Q_\alpha(t)\right)$

general solution: mode beating



general approach/case: $L = T - V = \frac{1}{2}\sum_{j,k=1}^{f}\left(T_{jk}\dot{q}_j\dot{q}_k - V_{jk}q_jq_k\right)$

$$\hookrightarrow \sum_{k=1}^{f}\left(V_{jk}-\omega^2T_{jk}\right)A_k = 0 \Rightarrow \det\left(V_{jk}-\omega^2T_{jk}\right)=0 \Rightarrow \omega_1^2,\dots,\omega_f^2$$

general solution: $q_k(t) = \mathrm{Re}\left\{\sum_{\alpha=1}^{f} C_\alpha A_k^{(\alpha)} e^{i\omega_\alpha t}\right\}$, $\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$

↑ C_α determined by initial conditions

3 Hamiltonian principle

3.1 variational problem

D'Alembert principle $\leftrightarrow$ differential variation (virtual displacements δq_i)

Hamiltonian principle $\leftrightarrow$ variation of entire trajectory

- Define an action functional $W: C^2 \to \mathbb{R}$

$$q(t) \mapsto W[q] := \int_{t_1}^{t_2} dt \, L(q, \dot{q}, t)$$

such that $\delta W[q] = 0$ for the real $q(t)$.

- variations $q'(t)$ of $q(t)$: $t_1 \le t \le t_2$

(i) $q'(t) \in C^2$ (ii) $\delta q(t) = q'(t) - q(t)$ (iii) $\delta t = 0$ (iv) $q'(t_1) = q(t_1)$, $q'(t_2) = q(t_2)$

3.2 Hamiltonian principle

- $$0 = \delta W[q] = \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \left\{ \frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \delta q_k \right\}$$

$$= \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \left\{ \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \right\} \delta q_k$$

$$\Rightarrow \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \quad k = 1, \dots, f \quad \text{(Euler-Lagrange equation)}$$

3.3 Gauge transformation

Theorem: The gauge transformation $L \to L' = L + \frac{d}{dt} M(q, t)$ leaves the Euler-Lagrange equations invariant.

4 Symmetries & laws of conservation

4.1 Noether-Theorem

idea: continuous infinitesimal transformation that leave the physical system invariant $\left(\frac{dL}{ds}=0\right) \Longleftrightarrow$ conserved quantity

Theorem: Let the Lagrange function/Lagrangian $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$ of an autonomous system be invariant w.r.t. infinitesimal transformations $q \longrightarrow \underline{h}^s(q)$ (s: continuous parameter, $\underline{h}^{s=0}(q)=q$)
Then there exists an integral of motion / conserved quantity / constant:

$$I(q,\dot{q}) = \sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} \left(\frac{d}{ds} h_j^s(q_j)\right)\bigg|_{s=0} .$$

- $\underline{h}^s(q) = \underbrace{\underline{h}^{s=0}(q)}_{=q} + s\left(\frac{d}{ds}\underline{h}^s(q)\right)\bigg|_{s=0} + O(s^2)$

4.2 Translational invariance

translational invariance $\longleftrightarrow$ conservation of momentum

$$\left.\begin{array}{l} \text{e.g.: } h^s: \underline{r}_i \rightarrow \underline{r}_i + s\,\underline{e}_x \\ \text{and } \dfrac{dL}{ds}=0 \end{array}\right\} \Rightarrow \sum_i m_i \dot{x}_i = P_x = \text{const.}$$

x-component of total momentum.

- generalized momenta: $p_j = \frac{\partial L}{\partial \dot{q}_j}$

- cyclic coordinates: $\frac{\partial L}{\partial q_j} = 0 \underset{\uparrow}{\Rightarrow} p_j = \text{constant}$

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}$$

4.3 Rotational invariance

Rotation w.r.t. z axis: $\underline{h}^s : \underline{r}_i \to \underline{r}_i' = \underline{h}^{s=0}(\underline{r}_i) + s \frac{d}{ds}\underline{h}^s(\underline{r}_i)\Big|_{s=0}$
(infinitesimal)

$$= \underline{r}_i + s\,(\underline{r}_i \times \underline{e}_z)$$

- invariant $L(\underline{r}_i, \dot{\underline{r}}_i) = L(\underline{r}_i', \dot{\underline{r}}_i)$ (rotation is an orthogonal transformation ⇒ only distance dependent)

$$\Rightarrow \left(\frac{dL}{ds}\right)_{s=0} = -\left(\frac{dV}{ds}\right)_{s=0} = -\sum_{i=1}^{N} \underbrace{\nabla_{\underline{r}_i} V}_{-\underline{F}_i} \cdot \underbrace{\frac{d\underline{r}_i'}{ds}\Big|_{s=0}}_{\underline{r}_i \times \underline{e}_z} = -\underline{e}_z \underbrace{\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i}_{\text{total torque}} \stackrel{!}{=} 0$$

Integral of motion

$$\Rightarrow I = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \left(\frac{dh^s}{ds}\right)_{s=0} = \dots = -\underline{e}_z \sum_{i=1}^{N} (\underline{r}_i \times m_i \dot{\underline{r}}_i) = -l_z$$

rotational invariance ⇒ conservation of angular momentum

- $\frac{\partial L}{\partial \varphi} = 0$ cyclic variable $\Leftrightarrow \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot\varphi}\right)}_{p_\varphi} = 0 \Leftrightarrow p_\varphi = \text{const}$

4.4 Temporally translational invariance

(i) no explicitly time-dependent constraint: $\underline{r}_i = \underline{r}_i(q_1, \dots, q_f)$

$$\frac{\partial \underline{r}_i}{\partial t} = 0 \Rightarrow \dot{\underline{r}}_i = \sum_{j=1}^{f} \frac{\partial \underline{r}_i}{\partial q_j} \dot q_j$$

(ii) $\frac{\partial L}{\partial t} = 0$ (scleronomic constraints)

$$\frac{d}{dt} L \underset{L = T(\dot q) - V(q)}{\overset{(1)}{=}} \dots = 2\frac{dT}{dt} \Rightarrow 0 = \frac{d}{dt}(2T - L) = \frac{d}{dt}(T+V)$$

$$\Rightarrow T + V = \text{const}$$

temporally translational invariance ⇒ conservation of energy

4.5 Two-body problem

2 (point) masses in 3D $\Rightarrow f=6$ degrees of freedom

$\Rightarrow 2f = 12$ integration constants / initial values

potential: $V(\underline{r}_1, \underline{r}_2) = V(|\underline{r}_1 - \underline{r}_2|)$ distance-dependent

(i) translational invariance $\Rightarrow$ conservation of momentum $\underline{P} = M\dot{\underline{R}} = \text{const}$

$\Rightarrow \underline{R}(t) = \frac{\underline{P}}{M} t + \underline{R}_0$

$\Rightarrow$ 3 initial values $\underline{P}$

3 — " — $\underline{R}_0$

(ii) rotational invariance $\Rightarrow$ conservation of angular momentum $\underline{l} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2 = \text{const}$

$\Rightarrow$ 3 initial conditions $\underline{l}$

(iii) temporally translational invariance $\Rightarrow$ conservation of energy

$$E = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 + V(|\underline{r}_1 - \underline{r}_2|) = \text{const}$$

$\Rightarrow$ 1 initial value E

$\Rightarrow$ 10 of 12 initial values given by system setup

coordinate of centre of mass: $\underline{R} = \frac{1}{M}(m_1 \underline{r}_1 + m_2 \underline{r}_2)$

relative coordinate : $\underline{r} = \underline{r}_1 - \underline{r}_2$

$$\left.\begin{array}{l} \underline{R} = \frac{1}{M}(m_1 \underline{r}_1 + m_2 \underline{r}_2) \\ \underline{r} = \underline{r}_1 - \underline{r}_2 \end{array}\right\} \Rightarrow \begin{array}{l} \underline{r}_1 = \underline{R} + \frac{m_2}{M} \underline{r} \\ \underline{r}_2 = \underline{R} - \frac{m_1}{M} \underline{r} \end{array}$$

$\Rightarrow$ Lagrangian: $L = \frac{M}{2} \dot{\underline{R}}^2 + \frac{m}{2} \dot{\underline{r}}^2 - V(r)$, $m = \frac{m_1 m_2}{m_1 + m_2}$ reduced mass

$\Rightarrow \underline{R}$ is acyclic variable $\Leftrightarrow \frac{\partial L}{\partial \dot{R}_k} = \text{const} = P_k \Rightarrow \underline{R}(t) = \frac{\underline{P}}{M} t + \underline{R}_0$

$\frac{\partial L}{\partial R_k} = 0$

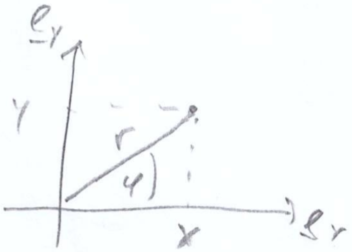
w.l.g. $\underline{P} = 0 = \underline{R}_0 \Rightarrow \underline{R} = 0$

angular momentum $\underline{l} = \ldots = m\,\underline{r}\times\dot{\underline{r}}$

$\Rightarrow \underline{l}\cdot\underline{r}=0\,,\quad \underline{l}\cdot\dot{\underline{r}}=0$ (motion in a plane)

$\Rightarrow$ effective 2D problem (w.l.g. $\underline{l}\parallel\underline{e}_z$

$\Rightarrow$ polar coordinates



$$\left.\begin{aligned} x &= r\cos\varphi \;\Rightarrow\; \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ y &= r\sin\varphi \;\Rightarrow\; \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{aligned}\right\} \Rightarrow \dot{\underline{r}}^2 = \dot{r}^2 + r^2\dot{\varphi}^2$$

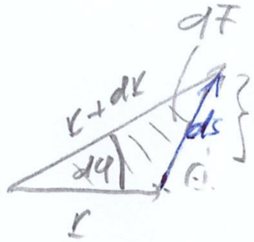
$$\Rightarrow L = \frac{m}{2}\left(\dot{r} + r^2\dot{\varphi}^2\right) - V(r)$$

$m(x\dot{y}-\dot{x}y)$

$$\Rightarrow \varphi \text{ is a cyclic variable} \Leftrightarrow \frac{\partial L}{\partial \varphi}=0 \qquad \frac{\partial L}{\partial\dot{\varphi}} = \text{const} = p_\varphi = m r^2\dot{\varphi} = l_z = l$$

$\underline{l}\parallel\underline{e}_z$

$\Rightarrow$ Kepler's 2^{nd} law: A line joining a planet and the sun sweeps out equal areas during equal intervals of time.



$|\underline{r}+d\underline{r}|\sin d\varphi$

$$dF = \frac{1}{2}|r|\,|r+dr|\sin d\varphi$$

$$\xrightarrow[dt\to0]{dr\to0} \frac{dF}{dt} = \frac{1}{2}r^2\frac{d\varphi}{dt} = \frac{1}{2}r^2\frac{l}{mr^2} = \frac{l}{2m} = \text{const}$$

alternative: $dF = \frac{1}{2}|\underline{r}\times d\underline{s}| = \frac{1}{2}|\underline{r}\times\dot{\underline{r}}\,dt| \Rightarrow \frac{dF}{dt} = \frac{1}{2}|\underline{r}\times\dot{\underline{r}}|$

$$= \frac{1}{2m}|\underline{r}\times m\dot{\underline{r}}|$$

$$= \frac{1}{2m}\, l$$

4.5.2 Conservation of energy & trajectory (around the sun / star)

Euler-Lagrange equation: $\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = 0$

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) - V(r)$$

$$\Rightarrow \quad m\ddot r - mr\dot\varphi^2 + \frac{\partial V}{\partial r} = 0 \quad , \quad \dot\varphi = \frac{l}{mr^2}$$

$$\Rightarrow \dots \Rightarrow \quad E = \frac{m}{2}\dot r^2 + \underbrace{\frac{l^2}{2mr^2} + V(r)}_{=:\tilde V(r)} = \text{const}$$

centrifugal potential $\frac{l^2}{2mr^2} \Rightarrow F_Z = \frac{l^2}{2mr^3}\frac{r}{r}$

$$\Rightarrow \quad t(r) - t_0 = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}(E - \tilde V(r))}} \quad \Rightarrow t(r) \text{ or } r(t)$$

$$\varphi(t) - \varphi_0 = \int_{t_0}^{t} dt' \frac{l}{mr^2(t')} \quad \Rightarrow \varphi(t)$$

$$\frac{dr}{d\varphi} = \frac{\dot r}{\dot\varphi} = \frac{mr^2}{l}\sqrt{\frac{2}{m}(E - \tilde V(r))}$$

$$\Rightarrow \varphi(r) - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{l^2}(E - \tilde V(r))}} \quad \Rightarrow \varphi(r),\ r(\varphi)$$

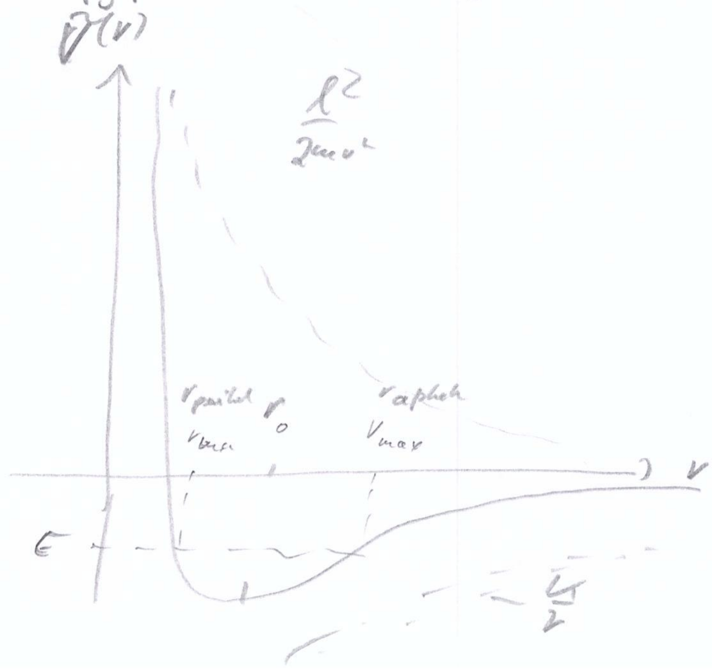
4.5.3 motion of planet & Kepler's laws

now: $V(r) = -\frac{\gamma m_1 m_2}{r} = -\frac{k}{r} \Rightarrow \tilde V(r) = -\frac{k}{r} + \frac{l^2}{2mr^2}$ $\begin{cases} \xrightarrow{r\to 0} \frac{l^2}{2mr^2} \to \infty \\ \xrightarrow{r\to\infty} -\frac{k}{r} \to 0 \end{cases}$

motion possible only for $E > \tilde V(r)$ $\quad (\frac{1}{2}m\dot r^2 = E - \tilde V)$

closed orbit $\quad -\frac{mk^2}{2l^2} < E < 0$

diverging path $\quad E > 0$



$$r_{min/max} = \frac{1}{2|E|}\left(k \pm \sqrt{k^2 - \frac{2l^2|E|}{m}}\right)$$

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{l^2}E + \frac{2mk}{l^2 r'} - \frac{1}{r'^2}}}$$

4.5.3 motion of planets & Kepler's laws

solving the integral $\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2}(E - \tilde{V}(r'))}}$

$$= \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2}\left(E + \frac{k}{r'} - \frac{\ell^2}{2mr'^2}\right)}}$$

by (i) quadratic extension and (ii) substitution

(i) $\frac{2m}{\ell^2}(E - \tilde{V}(r)) = \ldots = D\left[1 - \frac{1}{D}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)^2\right]$ with $D = \frac{2m}{\ell^2}\left(\frac{mk^2}{2\ell^2} + E\right)$

(ii) $\cos\vartheta'(r) = \frac{1}{\sqrt{D}}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)$

$\Rightarrow \frac{d}{dr'}\cos\vartheta'(r) = \frac{d}{dr'}\frac{1}{\sqrt{D}}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)$

$\Rightarrow -\sin\vartheta' \frac{d\vartheta'}{dr'} = -\frac{1}{\sqrt{D}\, r'^2}$ $\Rightarrow \frac{dr'}{\sqrt{D}\, r'^2} = +\sin\vartheta' d\vartheta'$

$\Rightarrow \varphi - \varphi_0 = \int_{\vartheta_0}^{\vartheta} d\vartheta' \sin\vartheta' \frac{1}{\sqrt{1 - \cos^2\vartheta'}} = \int_{\vartheta_0}^{\vartheta} d\vartheta' = \vartheta - \vartheta_0$

$\Rightarrow \varphi(r) = \arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)\right]$

$\Rightarrow r(\varphi) = \frac{\ell^2/mk}{1 + \varepsilon\cos\varphi}$

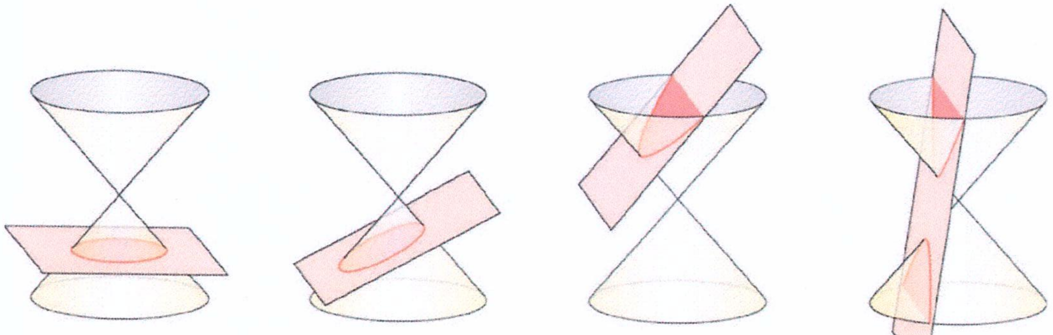
$\varepsilon = 0$: circle $0 < \varepsilon < 1$: ellipse $\left(-\frac{mk^2}{2\ell^2} < E < 0\right)$

$\varepsilon = 1$: parabola $\varepsilon > 1$: hyperbola

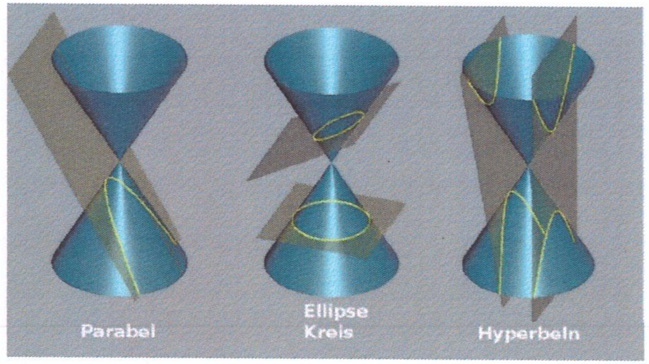
ε = numeric excentricity

Kepler's 3rd law: $T^2 \sim a^3$

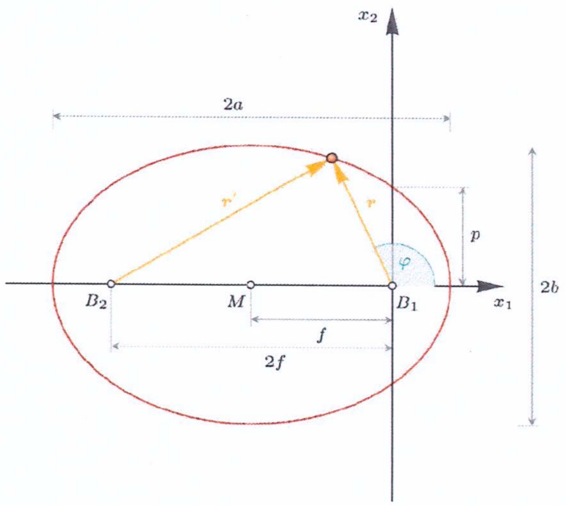
T: period a: semi axis of ellipse



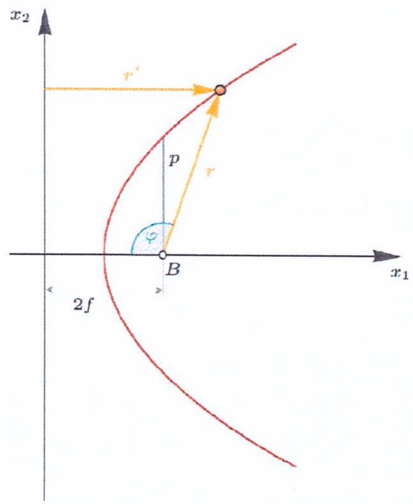
$\varepsilon = 0$ $0 < \varepsilon < 1$ $\varepsilon = 1$ $\varepsilon > 1$



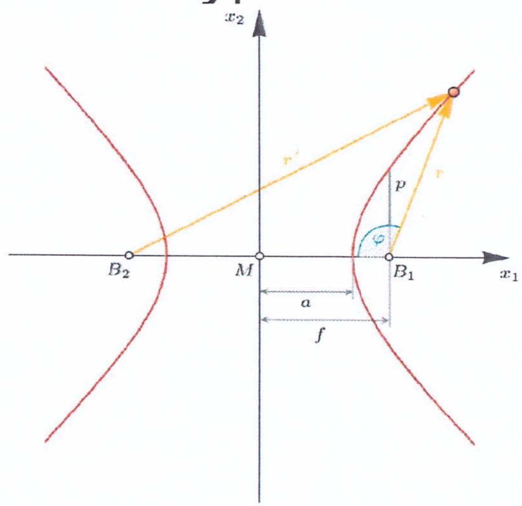
Ellipse



Parabel



Hyperbel



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5 Hamiltonformalism

idea: instead of $(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$ choose $(q_1, \dots, q_f, p_1, \dots, p_f)$

with $\frac{\partial L}{\partial \dot{q}_j} = p_j$

5.1 Legendre transformation

idea: $x \to u = \frac{df}{dx}$ $\Rightarrow$ inverse: $\varphi(u) = x$

$y = f(x)$

$$y \Rightarrow z = xu - f(x) = x\frac{df}{dx} - f(x) = \varphi(u)u - f(\varphi(u)) = g(u)$$

applied to Lagrangian:

$$y = L \quad , x = \dot{q} \Rightarrow u = \frac{\partial L}{\partial \dot{q}} = p \Rightarrow z = \dot{q}p - L(q,\dot{q},t) = H(q,p,t)$$

similarly: $H(q_1, \dots, q_f, p_1, \dots, p_f, t) = \sum_{j=1}^{f} \dot{q}_j p_j - L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$

5.2 Hamilton equations

calculate dH and equate term dq_j, dp_j, dt

$$\Rightarrow \quad \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$$

application protocol

(i) $q_1, \dots, q_f$

(ii) $\underline{r}_i = \underline{r}_i(q_1, \dots, q_f, t)$, $\dot{\underline{r}}_i = \dot{\underline{r}}_i(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$

(iii) $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$ $\Rightarrow$ (iv) $p_j = \frac{\partial L}{\partial \dot{q}_j}$

(v) $H = \sum_{j=1}^{f} \dot{q}_j p_j - L$

(vi) $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

5.3 Symplectic structure of the phase space

idea: find canonical transformations that leave the Hamiltonian equations $\dot{q}_j = \frac{\partial H(q_1,\dots,q_f,p_1,\dots,p_f,t)}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

form invariant: $(q,p) \longrightarrow (P,Q)$

$$H(q,p,t) \longrightarrow \overline{H}(P,Q,t)$$

- use Hamilton principle: $\delta \int_{t_1}^{t_2} dt\, L = \delta_{t_1} \int^{t_2} dt \{\sum_{j=1}^{f} p_j \dot{q}_j - H(q,p,t)\} = 0$

4 equivalent forms of generating functions of canonical transformations:

(i) $M_1(q,Q,t)$: $p_j = \frac{\partial M_1}{\partial q_j}$, $P_j = -\frac{\partial M_1}{\partial Q_j}$

$\overline{H} = H + \frac{\partial M_1}{\partial t}$

$$\Rightarrow \frac{\partial p_j}{\partial Q_k} = \frac{\partial^2 M_1}{\partial Q_k \partial q_j} = -\frac{\partial P_k}{\partial q_j}$$

(ii) $M_2(q,P,t)$: $p_j = \frac{\partial M_2}{\partial q_j}$, $Q_j = \frac{\partial M_2}{\partial P_j}$

$= M_1(q,Q,t) - \sum_{j=1}^{k} \underbrace{\frac{\partial M_1}{\partial Q_j}}_{-P_j} Q_j$

Legendre bzgl. Q_j

$$\Rightarrow \frac{\partial p_j}{\partial P_k} = \frac{\partial^2 M_2}{\partial P_k \partial q_j} = \frac{\partial Q_k}{\partial q_j}$$

(iii) $M_3(p,Q,t)$: $q_j = -\frac{\partial M_3}{\partial p_j}$, $P_j = -\frac{\partial M_3}{\partial Q_j}$

$= M_1 - \sum_{j=1}^{k} \underbrace{\frac{\partial M_1}{\partial q_j}}_{p_j} q_j$

Legendre bzg. q_j

$$\Rightarrow \frac{\partial q_j}{\partial Q_k} = -\frac{\partial^2 M_3}{\partial Q_k \partial p_j} = \frac{\partial P_k}{\partial p_j}$$

(iv) $M_4(p,P,t)$: $q_j = -\frac{\partial M_4}{\partial p_j}$, $Q_j = \frac{\partial M_4}{\partial P_j}$

Legendre bzgl. Q_j und q_j

$$\Rightarrow \frac{\partial q_j}{\partial P_k} = -\frac{\partial^2 M_4}{\partial P_k \partial p_j} = -\frac{\partial Q_k}{\partial p_j}$$

$$\underline{x} = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \quad \underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{1}} \\ -\underline{\underline{1}} & \underline{\underline{0}} \end{pmatrix}$$

$$\Rightarrow \quad \dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x \quad \Leftrightarrow \quad \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

- $\underline{\underline{J}}^2 = -\underline{\underline{1}}, \quad \underline{\underline{J}}^{-1} = \underline{\underline{J}}^T = -\underline{\underline{J}}$

$$\Rightarrow \quad \underbrace{-\underline{\underline{J}}}_{=\underline{\underline{J}}^T}\,\dot{\underline{x}} = \underline{H}_x$$

again the idea: transformations that leave $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

form invariant: $(q, p) \to (\underline{Q}, \underline{P})$

$$H(q,p,t) \to \bar{H}(Q, \underline{P}, t)$$

variable transform:
$$\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_f \\ p_1 \\ \vdots \\ p_f \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_f \\ P_1 \\ \vdots \\ P_f \end{pmatrix} \qquad \dot{\underline{x}} = \underline{\underline{J}}\, H_x$$

$$\Rightarrow \quad M_{\alpha\beta} = \frac{\partial x_\alpha}{\partial y_\beta}, \quad M^{-1}_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}$$

$$\Rightarrow \quad \underline{\underline{M}} = \underline{\underline{J}}\left(\underline{\underline{J}}\,\underline{\underline{M}}^{-1}\right)^T \Leftrightarrow \underline{\underline{M}}^T \underline{\underline{J}}\, \underline{\underline{M}} = \underline{\underline{J}}, \quad \underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \mathbb{1} \\ -\mathbb{1} & \underline{\underline{0}} \end{pmatrix}$$

- $\underline{\underline{J}}$ defines a skew-symmetric (non-degenerate, bilinear) form / alternating form: $(\underline{u}, \underline{v}) = -(\underline{v}, \underline{u}) = \underline{u}^T \underline{\underline{J}}\, \underline{v}$

- $(\underline{u}, \underline{u}) = 0 \Rightarrow 0 = \{\underline{u}+\underline{w}, \underline{u}+\underline{w}\} = \underbrace{\{\underline{u},\underline{u}\}}_{=0} + \{\underline{w},\underline{u}\} + \{\underline{u},\underline{w}\} + \underbrace{\{\underline{w},\underline{w}\}}_{=0}$

$$\Rightarrow \{\underline{w}, \underline{u}\} = -\{\underline{u}, \underline{w}\}$$

- $$\left.\begin{aligned} \dot{y}_j &= \sum_{k=1}^{2f} \frac{\partial y_j}{\partial x_k}\dot{x}_k \\ \frac{\partial H}{\partial y_j} &= \sum_{k=1}^{2f} \frac{\partial H}{\partial x_k}\frac{\partial x_k}{\partial y_j} \end{aligned}\right\} \quad \begin{aligned} \dot{\underline{y}} &= \underline{\underline{M}}^{-1}\dot{\underline{x}} = \ldots = \underline{\underline{J}}\,\underline{H}_y \quad (\dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x) \\ \underline{H}_y &= \underline{\underline{M}}^T \underline{H}_x \end{aligned}$$

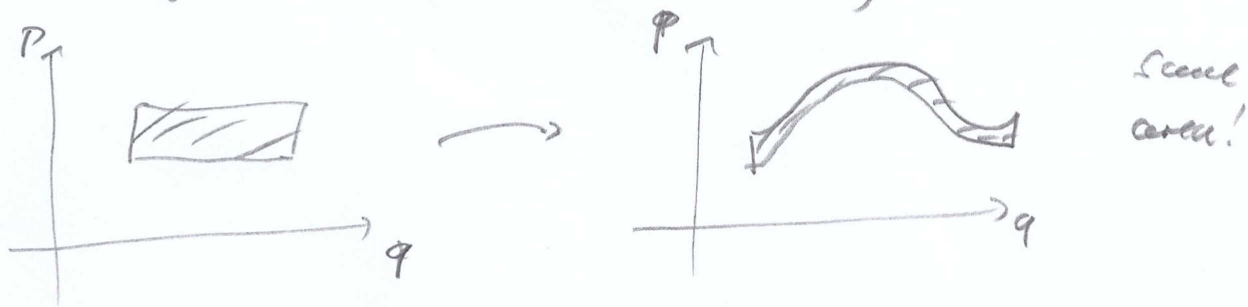
- $$\underline{\underline{M}} = \underbrace{\begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{pmatrix}}_{2f} \Big\} 2f \longleftrightarrow \begin{pmatrix} M_3(p,Q,t) & M_4(p,\underline{P},t) \\ M_1(q,Q,t) & M_2(q,\underline{P},t) \end{pmatrix}$$

Def: The set of matrices $\underline{\underline{M}}$ (canonical transforms) with $\underline{\underline{M}}^T \underline{\underline{J}}\, \underline{\underline{M}} = \underline{\underline{J}}$ forms the (real) symplectic group S over $\mathbb{R}^{2f}$.

5.4 Liouville's theorem

Thm: The flow of Hamiltonian systems preserves the phase space volume.

(The phase-space distribution function is constant along the trajectories of Hamiltonian systems.)



- flow: $\phi_{t,t_0} : \Gamma \to \Gamma$

$$\underline{x}_0(t_0) \mapsto \phi_{t,t_0}(x_0) = \underline{x}(t)$$

5.5 Poisson brackets

given 2 functions/observables $g(q,p,t)$ and $h(q,p,t)$, their Poisson brackets takes the form:
$$\{g,h\} = \sum_{i=1}^{f} \left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i} \right)$$

- $\{g,h\} = -\{h,g\}$
- $\{g, \lambda_1 h_1 + \lambda_2 h_2\} = \lambda_1 \{g, h_1\} + \lambda_2 \{g, h_2\}$
- $\{g,h\} = 0 \;\; \forall h \Rightarrow g = \text{const.}$
- $\{g,g\} = 0$
- $\frac{dg}{dt} = \{g, H\} + \frac{\partial g}{\partial t}$
- $\{g,h\} = (g_x, h_x) = \underline{g}_x^T \underline{\underline{J}}\, \underline{h}_x \;,\; \underline{g}_x = \begin{pmatrix} \frac{\partial g}{\partial q_1} \\ \vdots \\ \frac{\partial g}{\partial p_f} \end{pmatrix} ,\; \underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{1}} \\ -\underline{\underline{1}} & \underline{\underline{0}} \end{pmatrix} ,\; \underline{h}_x = \begin{pmatrix} \frac{\partial h}{\partial q_1} \\ \vdots \\ \frac{\partial h}{\partial p_f} \end{pmatrix}$

5.5 Poisson brackets (continued)

- $g = g(q, p, t) \Rightarrow \frac{dg}{dt} = \{g, H\}$
- fundamental Poisson brackets:

$$\left.\begin{array}{l}\{q_i, q_j\} = 0 = \{p_i, p_j\} \\ \{q_i, p_j\} = \delta_{ij}\end{array}\right\} \Leftrightarrow \{x_i, x_j\} = J_{ij}$$

$$\underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{1}} \\ -\underline{\underline{1}} & \underline{\underline{0}} \end{pmatrix}$$

- Theorem: The transformation $(q, p) \to (\underline{Q}, \underline{P})$ is canonical

  iff $\{Q_i, P_j\} = \delta_{ij}$, $\{Q_i, Q_j\} = 0 = \{P_i, P_j\}$.

  → Proof?

- wrap up: equivalent statements:

  (i) $\underline{x} \mapsto \underline{y}$ is canonical

  (ii) $\dot{\underline{x}} = \underline{\underline{J}}\, \underline{H}_x$ are invariant.

  (iii) $\{g, h\}$ are invariant

  (iv) $\{x_i, x_j\} = J_{ij}$ are invariant

  (v) Jacobian $M_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}$ is symplectic $(\underline{\underline{M}}^T \underline{\underline{J}}\, \underline{\underline{M}} = \underline{\underline{J}})$

  (vi) There exists a generating function

- link to QM

5.6 Hamilton-Jacobi theory

idea: find canonical transform for which all variables become canonical.

$\hookrightarrow \bar{H} \equiv 0$

$\hookrightarrow F_2(q, \underline{P}, t) =: S(q, \underline{P}, t)$ with: $(q, p) \rightarrow (\underline{Q}, \underline{P})$

$$H(q,p,t) \rightarrow \bar{H}(\underline{Q}, \underline{P}, t) = H + \frac{\partial S}{\partial t}$$

$$p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial P_k}$$

$$\Rightarrow \bar{H} = H\left(q_1, \dots, q_f, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_f}, t\right) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k} = 0 \quad \Rightarrow Q_k = \beta_k = \text{const.}$$

"parameter"

$$\dot{P}_k = -\frac{\partial \bar{H}}{\partial Q_k} = 0 \quad \Rightarrow P_k = \alpha_k = \text{const.} \Rightarrow S(q, \underline{\alpha}, t)$$

- recipe:

(i) $H(q,p,t)$, $p_k = \frac{\partial S}{\partial q_k} \Rightarrow H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$

(ii) Solve Hamilton-Jacobi equation $\Rightarrow S(q, \underline{\alpha}, t)$; $\underline{\alpha} = \underline{P}$

(iii) $Q_k = \frac{\partial S}{\partial \alpha_k} = \beta_k \Rightarrow q_j = q_j(\underline{\alpha}, \beta, t)$

(iv) $p_j = \frac{\partial S}{\partial q_j} = p_j(q, \underline{\alpha}, t)$

(v) Determine $\underline{\alpha}, \beta$ from initial conditions:

$$\left.\begin{array}{l} q_j(0) = q_j(\underline{\alpha}, \beta, 0) \\ p_j(0) = p_j(\underline{\alpha}, \beta, 0) \end{array}\right\} \Rightarrow \begin{array}{l} \underline{\alpha} = \underline{\alpha}(q(0), p(0)) \\ \beta = \beta(q(0), p(0)) \end{array}$$

- $S = \int L \, dt$ "action"
- $\frac{\partial H}{\partial t} = 0 \Leftrightarrow \frac{dH}{dt} = \{H, H\} = 0 \Rightarrow H$: Integral of motion $\Rightarrow S(q, P, t) = W(q) - Et$

$H(q, ~~\dots~~) = E$

6. Mechanics of a rigid body

idea: spatially extended system of

(a) N masses with fixed distances $M = \sum_{i=1}^{N} m_i$

(b) continuous density of mass $M = \int d^3r\, \rho(\underline{r})$

coordinates:

(i) labsystem (system of inertia) : S

(ii) body-fixed coordinatesystem : S^*

(iii) translational system of (i) with center of mass as origin : S'

position in S^*: $\underline{r}^* = \underline{d} = \underset{\text{position}}{\underline{r}} - \underset{\text{center of mass}}{\underline{r}_0}$

$\Rightarrow |\underline{d}| = \text{const}$, but orientation can change

- Euler angles: φ, ϑ, ψ

φ: rotation around $\underline{e}_3$ $\rightarrow \underline{\underline{R}}_3(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

ϑ: rotation around $\underline{e}_1^{(\varphi)}$ $\rightarrow \underline{\underline{R}}_1(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}$

ψ: rotation around $\underline{e}_3^{(\vartheta)}$ $\rightarrow \underline{\underline{R}}_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ combined/chained rotation: $\underline{\underline{R}}(\varphi, \vartheta, \psi) = \underline{\underline{R}}_3(\psi)\, \underline{\underline{R}}_1(\vartheta)\, \underline{\underline{R}}_3(\varphi)$

6.1 kinetic energy & inertia tensor

(a) $T = \underbrace{\frac{1}{2} M \dot{\underline{r}}_0^2}_{\text{center of mass}} + \underbrace{\frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}\, \underline{\omega}}_{\text{rotation}}$

$\Theta_{jk} = \sum_{i=1}^{N} m_i \left[(d^{(i)})^2 \delta_{jk} - d_j d_k \right]$

(b) $T = \frac{1}{2} M \dot{\underline{r}}_0^2 + \frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}\, \underline{\omega}$

$\Theta_{jk} = \int d^3r\, \rho(\underline{d}) \left[d^2 \delta_{jk} - d_j d_k \right]$

Θ of Sphere

$\rho(r) = \begin{cases} M / \frac{4}{3}\pi R^3 & r \leq R \\ 0 & r > R \end{cases}$

$\Theta = \Theta_1 = \Theta_2 = \Theta_3 = \frac{2}{5} M R^2$

$\underline{\underline{\Theta}} = \begin{pmatrix} \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \Theta \end{pmatrix}$

6.2 Properties of inertia tensor (continued)

(a) $\Theta_{jk} = \sum_{i=1}^{N} m_i \left[d^{(i)2} \delta_{jk} - d_j^{(i)} d_k^{(i)} \right]$

(b) $\Theta_{jk} = \int d^3r \; \rho(\underline{d}) \left(d^2 \delta_{ij} - d_j d_k \right)$

- diagonalization of $\underline{\underline{\Theta}}$: $\underline{x} \rightarrow \underline{x}' = \underline{\underline{R}}\, x$

$$\underline{\underline{\Theta}} \rightarrow \underline{\underline{\Theta}}' = \underline{\underline{R}}^T \underline{\underline{\Theta}}\, \underline{\underline{R}}$$

such that $\underline{\underline{\Theta}}' = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix}$ $\quad$ Θ_i principal moments of inertia

$\underline{\omega}^{(i)}$: principal axis

$\Rightarrow$ eigenvalue equation: $\underline{\underline{\Theta}}\, \underline{\vec{\omega}}^{(i)} = \Theta_i \underline{\omega}^{(i)}$

$\Theta_1 = \Theta_2 = \Theta_3$ spherical top (symmetry, not necessarily a sphere!)

$\Theta_1 = \Theta_2 \neq \Theta_3$ symmetric top

$\Theta_1 \neq \Theta_2 \neq \Theta_3$ asymmetric top

- moment of inertia w.r.t. $\underline{\omega} = \underline{\vec{n}}\, \omega$: $\Theta_n = \underline{\vec{n}}^T \underline{\underline{\Theta}}\, \underline{\vec{n}}$

$$= \sum_{i=1}^{N} m_i \left[d^{(i)2} - (\underline{d} \cdot \underline{\vec{n}})^2 \right]$$

$$= \sum_{i=1}^{N} m_i \left(\ell^{(i)} \right)^2$$

$$\hat{=}\; \underline{d}^{(i)} - \underline{\vec{n}} (\underline{d}^{(i)} \cdot \underline{\vec{n}})$$

$\Rightarrow T_{rot} = \frac{1}{2} \Theta_n \omega^2$

- Steiner's theorem: $\Theta'_{jk} = \Theta_{jk} + M \left(a^2 \delta_{jk} - a_j a_k \right)$

with rotation axes of S and S' shifted by $\underline{a}$

- $\Theta_n' = \Theta_n + M \ell^2$, $\ell^2 = a^2 - (\underline{a} \cdot \underline{\vec{n}})^2$, $\Theta_n = \underline{\vec{n}}^T \underline{\underline{\Theta}}\, \underline{\vec{n}}$

6.3 Angular momentum & equations of motion

(a) $$\underline{L} = \underbrace{M \underline{r}_0 \times \dot{\underline{r}}_0}_{\underline{L}_s\text{: center of mass}} + \underbrace{\sum_{i=1}^{N} m_i \, \underline{d}^{(i)} \times \left(\underline{\omega} \times \underline{d}^{(i)}\right)}_{\underline{L}_R\text{: relative angular momentum}}$$

(b) $$\underline{L} = M \underline{r}_0 \times \dot{\underline{r}}_0 + \int d^3r \, \rho(\underline{d}) \, \underline{d} \times (\underline{\omega} \times \underline{d})$$

$$\Rightarrow \underline{L}_R = \underline{\underline{\Theta}} \, \underline{\omega}$$

in principle: $\underline{L}_R \nparallel \underline{\omega}$ (rotation by $\underline{\underline{\Theta}}$)

6.3 Angular momentum & equations of motion

Center of mass theorem: $M\ddot{\underline{r}}_0 = \sum_{i=1}^{N} \underline{F}^{(i)\,\text{external}} = \underline{F}^{\text{external}}$

Angular momentum theorem: $\sum_{i=1}^{N} m_i\, \underline{d}^{(i)} \times \ddot{\underline{d}}^{(i)} = \sum_{i=1}^{N} \underline{d}^{(i)} \times \underline{F}^{(i)\,\text{external}}$

$$\frac{d}{dt}\underline{L} = \underline{\underline{M}} \quad (\text{torque})$$

- $\frac{d}{dt}\underline{L} = \frac{d^*}{dt^*}\underline{L} + \underline{\omega} \times \underline{L}$

  ↑ body-fixed derivative

- Euler's equations: $\frac{d}{dt}\underline{L} = \underline{\underline{\Theta}}\,\dot{\underline{\omega}} + \underline{\omega} \times \underline{\underline{\Theta}}\,\underline{\omega} = \underline{M}$

$$\overset{L_i = \Theta_i \omega_i}{\Longleftrightarrow} \left\{ \begin{aligned} &\Theta_1 \dot{\omega}_1 + (\Theta_3 - \Theta_2)\, \omega_2 \omega_3 = M_1 \\ &\Theta_2 \dot{\omega}_2 + (\Theta_1 - \Theta_3)\, \omega_3 \omega_1 = M_2 \\ &\Theta_3 \dot{\omega}_3 = (\Theta_2 - \Theta_1)\, \omega_1 \omega_2 = M_3 \end{aligned} \right.$$

coupled, nonlinear eqs.

6.4 Examples:

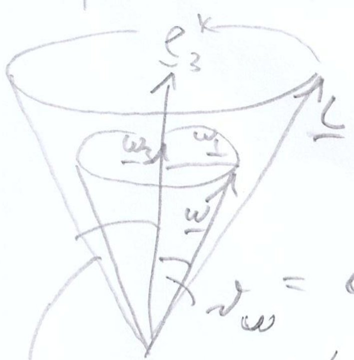
1) Torque-free top (symmetric: $\Theta_1 = \Theta_2 = \Theta \neq \Theta_3$)

$\Rightarrow \omega_3 = \text{const}$

$\omega_1(t) = \omega_\perp \sin(\omega_0 t - \varphi_0)$, $\quad \omega_0 = \frac{\Theta - \Theta_3}{\Theta}\, \omega_3$

$\omega_2(t) = -\omega_\perp \cos(\omega_0 t - \varphi_0)$

$\omega_1^2 + \omega_2^2 = \omega_\perp^2$



$\vartheta_\omega = \arctan \frac{\omega_\perp}{\omega_3}$ (Nile)

$\vartheta_{\text{precession}} = \arctan \frac{L_\perp}{L_3} = \arctan \frac{\Theta \omega_\perp}{\Theta_3 \omega_3} = \vartheta$ (Euler angle)

System of inertia / lab frame:

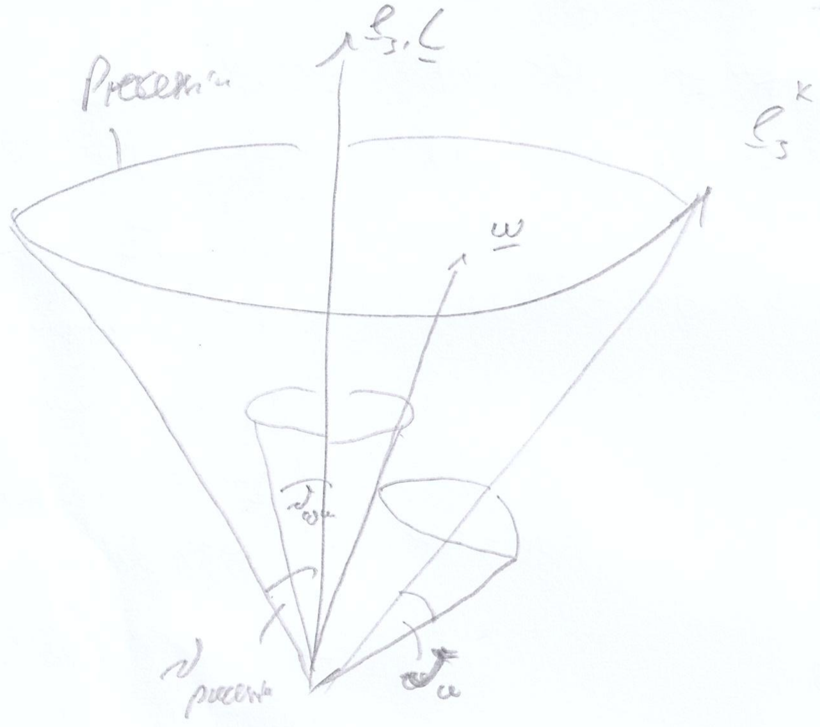
$$\begin{pmatrix}\omega_1^k\\ \omega_2^k\\ \omega_3^k\end{pmatrix} = \dot\varphi\begin{pmatrix}\sin\psi\sin\vartheta\\ -\cos\psi\sin\vartheta\\ \cos\vartheta\end{pmatrix} + \dot\psi\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \dot\vartheta = 0$$

$$= \begin{pmatrix}\omega_\perp^k \sin(\Omega t)\\ \omega_\perp^k\cos(\Omega t)\\ \mathrm{const}\end{pmatrix}$$

$\Rightarrow$ Euler angles: $\psi = \Omega t + \psi_0$, $\dot\varphi = \Omega' = \dfrac{\Theta_3\,\omega_3^k}{\Theta\cos\vartheta}$, $\vartheta(t) = \vartheta_0$

$\dot\psi = \Omega$ $\qquad \Rightarrow \varphi = \Omega' t + \varphi_0$

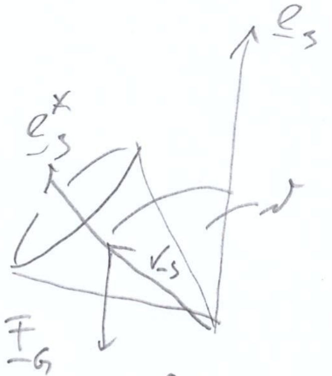
- $\sphericalangle(\underline{e}_3, \underline{e}_3^k) = \vartheta_{\mathrm{cone}\,L} = \arctan\dfrac{\Omega\sin\vartheta}{\Omega' + \Omega\cos\vartheta}$



6.4 Examples:

(1) torque-free top (symmetric)

(2) symmetric top subject to gravity



$\underline{r}_s = R\,\underline{e}_3^*$ center of mass, $\underline{F}_G = -mg\,\underline{e}_3$

$\Rightarrow$ Torque: $\underline{M} = \underline{r}_s \times \underline{F}_G = -mgR\,\underline{e}_3^* \times \underline{e}_3$

$\Rightarrow \underline{M} \perp \underline{e}_3^* \wedge \underline{M} \perp \underline{e}_3$

total energy: $E = T + V = \frac{1}{2}\Theta(\omega_1^2 + \omega_2^2) + \frac{1}{2}\Theta_3\omega_3^2 + mgR\cos\vartheta$

$= \text{const}$

$\hookrightarrow \dot{u}^2 + V_{eff}(u) = 0$

motion with $-1 < u_1 \le \cos\vartheta \le u_2 < 1$ Nutation

(3) torque-free asymmetric: $0 < \Theta_1 < \Theta_2 < \Theta_3$

$$\left.\begin{aligned} \dot{\omega}_1 &= -\alpha_1 \omega_2 \omega_3 \\ \dot{\omega}_2 &= \alpha_2 \omega_3 \omega_1 \\ \dot{\omega}_3 &= -\alpha_3 \omega_1 \omega_2 \end{aligned}\right\} \alpha_i > 0 \qquad \alpha_1 = \frac{\Theta_3 - \Theta_2}{\Theta_1}, \quad \alpha_2 = \frac{\Theta_3 - \Theta_1}{\Theta_2}, \quad \alpha_3 = \frac{\Theta_2 - \Theta_1}{\Theta_3}$$

fixed points: $\underline{\omega}^* \in \left\{ \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \right\}$

$\Rightarrow$ linearization: $\delta\underline{\omega} = \underline{\omega} - \underline{\omega}^*$:

$$\begin{pmatrix} \delta\dot{\omega}_1 \\ \delta\dot{\omega}_2 \\ \delta\dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_1 \omega_3 & -\alpha_1 \omega_2 \\ \alpha_2 \omega_3 & 0 & \alpha_2 \omega_1 \\ -\alpha_3 \omega_2 & -\alpha_3 \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \end{pmatrix}$$

$\Rightarrow$ eigenvalues:

		(not unstable)
$(\omega,0,0)$	$\lambda_1 = 0$, $\lambda_{2,3} = \pm i\omega\sqrt{\alpha_2\alpha_3}$	stable / center
$(0,\omega,0)$	$\lambda_1 = 0$, $\lambda_{2,3} = \pm\omega\sqrt{\alpha_1\alpha_3}$	unstable / saddle
$(0,0,\omega)$	$\lambda_1 = 0$, $\lambda_{2,3} = \pm i\omega\sqrt{\alpha_1\alpha_2}$	stable / center

$\Rightarrow$ 1 unstable motion / rotation around axis relating to Θ_2

7. Continuum mechanics

7.1 3 spring coupled oscillators

$$L = T - V = \frac{m}{2}\sum_{i=0}^{N+1} \dot{q}_i^2 - \frac{D}{2}\sum_{i=0}^{N+1}(q_i - q_{i-1})$$

equation of motion: $m\ddot{q}_j - D(q_{j-1} - 2q_j + q_{j+1}) = 0 \quad j = 1,2,3.$

boundary conditions: $q_0(t) = 0 = q_{N+1}(t)$

solution / ansatz: $q_j(t) = C a_j \cos(\omega t - \delta)$

$$\hookrightarrow \begin{pmatrix} 2\frac{D}{m} & -\frac{D}{m} & 0 \\ -\frac{D}{m} & 2\frac{D}{m} & -\frac{D}{m} \\ 0 & -\frac{D}{m} & 2\frac{D}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$ eigenvalue equation

$$\Rightarrow \lambda_1 = \omega_1^2 = 2\frac{D}{m} \Rightarrow \omega_1 = \sqrt{2}\,\omega_0, \quad \omega_0 = \frac{D}{m} \quad \underline{a}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_{2,3} = \omega_{2,3}^2 = (2 \pm \sqrt{2})\frac{D}{m} \Rightarrow \omega_2 = \sqrt{2 - \sqrt{2}}\,\omega_0, \quad \underline{a}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\omega_3 = \sqrt{2 + \sqrt{2}}\,\omega_0, \quad \underbrace{\underline{a}^{(3)} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}}_{\text{on sine curves!}}$$

7.2 limit to continuum

$q_j(t) = C \sin(\alpha j)\cos(\omega t - \delta), \quad j = 1, \dots, N, \quad q_0(t) = 0 = q_{N+1}(t)$

with $\alpha = \frac{\pi k}{N+1}$

$$\hookrightarrow \left[-m\omega^2 + 2D\left(1 - \cos\left(\frac{\pi k}{N+1}\right)\right)\right] C \sin(\alpha j) = 0$$

$$\hookrightarrow \omega_k = 2\sqrt{\frac{D}{m}}\sin\left(\frac{\pi k}{2(N+1)}\right)$$ checked for $k = 1,2,3$

$$\Rightarrow q_j(t) = \sum_{k=1}^{N} \sin\left(\frac{\pi k}{N+1} j\right) \underbrace{C_k \cos(\omega_k t - \delta_k)}_{[d_k \cos(\omega_k t) + e_k \sin(\omega_k t)]}$$

limits:

$N \to \infty$, $l_0 \to 0$: $(N+1)\, l_0 = l = \text{const}$ (total length)

$m \to 0$, $l_0 \to 0$: $\frac{m}{l_0} = \rho = \text{const}$ (density of mass)

$D \to \infty$, $l_0 \to 0$: $D l_0 = \text{const}$ (total potential energy)

$= P = Y$ (modul of elasticity)

with spatial coordinate: $x = j\, l_0 \to q_j(t) \to q(x,t)$

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} q(x,t) = \frac{\partial^2}{\partial x^2} q(x,t)$$ wave equation

$$\frac{1}{c^2} = \frac{m}{l_0} \frac{1}{D l_0}$$

7.3 Wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,t) = \frac{\partial^2}{\partial x^2}\psi(x,t) \qquad c\text{: (phase) velocity}$$

↳ D'Alembert | ansatz solution $\psi(x,t) = f(x \pm ct)$

↳ standing waves in co-moving frame

↳ Bernoulli | solution: ansatz $\psi(x,t) = g(x)\,h(t)$

$$\Rightarrow \ddot{h}(t) = -c^2k^2\,h(t) \quad \text{and} \quad g''(x) = -k^2 g(x)$$

e.g. (i) guitar string:

$$\psi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}x\right)\left(d_n\cos(\omega_n t) + e_n \sin(\omega_n t)\right)$$

(ii) 2D rectangular | membrane drum : $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,y,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,t)$

$$\hookrightarrow \psi(x,y,t) = \sum_{n,s=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{s\pi}{b}y\right)\left[d_{ns}\cos(\omega_{ns}t) + e_{ns}\sin(\omega_{ns}t)\right]$$

$$\omega_{ns} = \pi c\sqrt{\frac{n^2}{a^2} + \frac{s^2}{b^2}}$$

$+ f(x,t)\,\psi(x,t)$

• Limit of Lagrange function:

$$L = T - V = \int_0 dx \underbrace{\left[\frac{\rho}{2}\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2 - \frac{P}{2}\left(\frac{\partial\psi}{\partial x}\right)^2\right]}_{\mathcal{L}\text{ : Lagrange density}}$$

$$\hookrightarrow \delta S = \delta\int_{t_1}^{t_2} dt\, L = 0$$

$$\Rightarrow \frac{\partial}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\psi'} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \qquad \left(\text{cf. } \frac{d}{dt}\frac{\partial L}{\partial\dot{q}_j} - \frac{\partial L}{\partial q_j} = 0\right)$$

$f(x,t)$ in case of external force

total force $F(t) = \int_l dx\, f(x,t)$

7.3 Wave equation

- circular/annular membrane

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt} \qquad \text{with} \quad u(r=l, \varphi, t) = 0 \ \text{(boundary condition)}$$

protocol: separation of (i) t and (r,φ) dependence

(ii) r and φ dependence

$$\Rightarrow u(r,\varphi,t) = R(r)\,\phi(\varphi)\,T(t) \quad , \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

(i) $\ddot{T} + c^2 k^2 T = 0 \quad \Rightarrow T(t) = A\cos(\omega t) + B\cos(\omega t)$

(ii) $\frac{d^2\phi}{d\varphi^2} + \lambda^2 \phi = 0 \quad \Rightarrow \phi(\varphi) = C\cos(\lambda\varphi + \varphi_0)$ with $\lambda \in \mathbb{Z}$ (2π periodicity)

(iii) $r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (k^2r^2 - \lambda^2)R = 0$ Bessel equation $(x = kr)$

Ansatz: $R(x) = x^{\beta} \sum_{s=0}^{\infty} \alpha_s x^s$

$$\Rightarrow R(r) = \sum_{m=0}^{\infty} D_m J_m(kr) \quad \text{with } J_m \text{ Besselfunktion}$$

$$\Rightarrow u(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \left[A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\right]\cos(m\varphi + \varphi_0)\, J_m(k_{mn}r)$$

with $k_{mn} = \frac{\mu_{mn}}{l}$, μ_{mn}: n-th root of J_m (boundary conditions)

- Lagrange density in 3D: $L = \int_V d^3r \left[\frac{\rho}{2}\, \dot{\underline{q}}(r,t)^2 - V(\underline{q}, \underline{\nabla}\circ\underline{q})\right]$

density of potential energy

$$V = V^{\text{internal}} + \underbrace{V^{\text{external}}}_{-\underline{f}(r,t)\cdot\underline{q}(r,t)}$$

$$V^{\text{internal}} \approx V^{\text{internal}}(0) + \sum_{j=1}^{3} \frac{\partial V^{\text{internal}}}{\partial \eta_{ij}}\eta_{ij} + \frac{1}{2}\sum_{ijkl}^{3} \underbrace{\frac{\partial^2 V^{\text{internal}}}{\partial \eta_{ij}\partial \eta_{kl}}}_{\substack{C_{ijkl} \\ \text{elasticity tensor}}} \eta_{ij}\eta_{kl}$$

$$\underline{\underline{\eta}} = \underline{\nabla}\circ\underline{q} = \begin{pmatrix} \partial_x q_x & \partial_x q_y & \partial_x q_z \\ \partial_y q_x & \partial_y q_y & \partial_y q_z \\ \partial_z q_x & \partial_z q_y & \partial_z q_z \end{pmatrix}$$