

Theoretische Physik Ib: Analytische Mechanik 7

VL SS2025, Dr. habil. Philipp Hövel

philipp.hoewel@uni-saarland.de, E2 6, Raum 4.03

VL: Di: 12-14 c.t. (12:15-13:45) E2 5, Hörsaal II/02
Do: 12-14 c.t. (12:15-13:45) E2 5, Hörsaal II/02 } 4h $\Rightarrow$ 6h

UE: Mi: 14:15-15:45 E2 6, 4.15, Start: 16.4.
Fr: 10:15-11:45 E2 6, 1.14 } Start: 11.4.
Fr: 12:15-13:45 E2 6, 2.21 }
(pünktliche Schluss)

TUT: Do: 14-16 (2 13:45-15:15) E2 5, Hörsaal II/02
optionales Angebot
6h $\Rightarrow$ 9h

Umfang: - 8 ECTS (B.Sc. Physik ...) $\Rightarrow$ 240h $\rightarrow$ 150h

Zeit außerhalb der Vorlesungstf. $\Rightarrow$ 10h/Woche

- Option b für Lehramt

Note: Klausurnote

Übungsblätter $\xrightarrow{\geq 50\%}$ Klausur $\xrightarrow{\geq 50\%}$ Bestanden
12 Blätter
(Di $\rightarrow$ Di)

Start Do: 10.4.

Stil: anspruchsvoll orientiert, anregend, schülerorientiert, aufbauend,
viele Hinweise rechts, links, oben, unten, vorne, hinten...

"Bergföhneffekt" & "Lärche"

Hilfsmittel: - VL-Notizen

- Glossar (individuell)

- Definitionen, Sätze ... formal

- Lehrbücher (s. Koolke)

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0. Übersicht

Klassische Mechanik: • Bewegung von Körpern (**deteterministisch**, d.h. aus **Anfangsbedingungen** vorher berechenbar;
(vgl. Relativistische Mechanik, Quantenmechanik, Statistische Mechanik) **Kausal**, d.h. verursacht durch **Kräfte**)

• Grundlage der theoretischen Physik

↳ physikalische Grundbegriffe

↳ Paradigmen einer physikalischen Theorie

(mathematische - geometrische Strukturen der Dynamik)

• Fokus: - Symmetrien & Invarianzprinzipien

- geometrische Strukturen

- Nichtlineare Theorie

- Grundlage für andere Theorien

- Verallgemeinerte **kanonische** Formulierung

↳ **Lagrange** → Feldtheorien (z.B.: E-Dynamik, Relativitätstheorie)

↳ **Hamilton** → Quantenmechanik, statistische Mechanik

- Differenzialprinzip: **D'Alembert**

- Integralprinzip: **Hamilton**

1. Newtonsche Mechanik

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1. Herangehensweise
2. Newtonsche Axiome
3. Galilei-Transformation
4. Weitere Folgerungen

1. Newtonsche Mechanik

1.1 Herangehensweise (des Pädels Kern)

Kinematik & Dynamik von Systemen von Massepunkten
ohne Zwangsbedingungen

1.2 Newtonsche Axiome

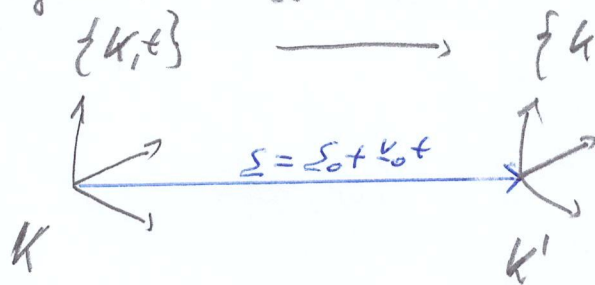
- (I) Kräftefrei Körper bewegen sich geradlinig und gleichförmig
(lex prima / Trägheitsgesetz)
- (II) Beschleunigung: $\underline{a} := \frac{d}{dt} \underline{v} \sim \underline{F}$ *Newton* $(\underline{F} = \dot{\underline{p}} \stackrel{m=\text{const.}}{=} m \underline{a})$
(lex secunda / Bewegungsgesetz)
- (III) *actio = reactio*
(lex tertia / Reaktionsgesetz)
- (IV) Lineare Superposition von Kräften
(lex quarta / Superpositions-gesetz)

- Bem.: - Raum ist *homogen* & *isotrop* (3D, Euklidischer Raum) } nicht-relativistisch
- Zeit ist universell ("universelle" Sequenzgeschehnisfolge)
- Ereignis: $(\underline{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$
- dynamische Variable $\underline{r}(t)$: Bahnkurve / Trajektorie
 $\underline{v}(t) := \frac{d}{dt} \underline{r}(t) \equiv \dot{\underline{r}}(t)$ *Geschwindigkeit* } *(Tangentenvektor)*
- 1. Axiom $\Rightarrow$ *Inertialsystem* (def. durch kräftefreie Bewegung)
 $\hookrightarrow$ Folgerung: Galilei-Transformation

1.3 Galilei-Transformationen (Folgerung aus (I))

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- allgemeine Transformation zwischen 2 Inertialsystemen



$$\underline{r}(t) = \underline{R} \underline{r}'(t) + \underline{v}_0 t + \underline{x}_0 \quad \text{mit } \underline{R} : \text{Drehmatrix}$$

$$t = t' + t_0$$

$$\underline{\dot{r}}(t) = \underline{\dot{r}}'(t') + \underline{v}_0$$

Newton'sche Mechanik ist **forminvariant** gegenüber den **Galilei-Transformationen**
 ↳ **Galilei invariant**

1.4 Weitere Folgerungen

(II): $\underline{a} \sim \underline{F}$: Proportionalitätsfaktor: m : träge Masse

⇒ **Bewegungsgleichung**: $\underline{F}(\underline{r}, \underline{\dot{r}}, t) = m \frac{d^2 \underline{r}}{dt^2}$ (3 gekoppelte i.a. nichtlineare DGLs)
 eindeutige Lösung zu Anfangsbedingungen $(\underline{r}_0, t_0) : \underline{r}(t; \underline{r}_0, t_0)$

$$(III) \quad \underline{F}^{(12)} + \underline{F}^{(21)} = 0 \quad (\text{Achtung: Notation vertauschen in Büchern})$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{Kraft} & \text{Kraft} \\ \text{von 2 auf 1} & \text{von 1 auf 2} \end{array} \quad \left(\text{Matrixnotation} \quad \begin{pmatrix} \underline{F}^{(11)} & \underline{F}^{(12)} \\ \underline{F}^{(21)} & \underline{F}^{(22)} \end{pmatrix} \begin{pmatrix} \text{Teilchen 1} \\ \text{Teilchen 2} \end{pmatrix} \right)$$

⇒ Same äußere Kräfte:

$$\left. \begin{array}{l} m^{(1)} \underline{a}^{(1)} = m^{(1)} \frac{d \underline{v}^{(1)}}{dt} = \underline{F}^{(11)} \\ m^{(2)} \underline{a}^{(2)} = m^{(2)} \frac{d \underline{v}^{(2)}}{dt} = \underline{F}^{(21)} \end{array} \right\} \frac{d}{dt} \left(\underbrace{m^{(1)} \underline{v}^{(1)}}_{\underline{p}^{(1)}} + \underbrace{m^{(2)} \underline{v}^{(2)}}_{\underline{p}^{(2)}} \right) = \underline{F}^{(12)} + \underline{F}^{(21)} = 0$$

⇒ Erhaltung des Gesamtimpulses $\underline{p}^{(1)} + \underline{p}^{(2)} = \text{const.}$

(IV) Vektorcharakter der Kräfte $\Rightarrow$ **Superposition**

Kräfte $\sim$ Felder $\rightarrow$ lineare Feld-Theorien (z.B. E-Dynamik)

oder: Bewegungsgleichung $\vec{F}(\vec{v}) = m \frac{d\vec{v}}{dt}$ i. A. nichtlinear in $\vec{v}$
(Ausnahme: harmonischer Osz.)
 $\vec{F}(\vec{v}) \sim \vec{v}$

oder: **Gravitationskraft** $\vec{F}^{(12)} = -G m_s^{(1)} m_s^{(2)} \frac{\vec{r}^{(1)} - \vec{r}^{(2)}}{|\vec{r}^{(1)} - \vec{r}^{(2)}|^3}$

$m_s^{(i)}$: Schwere Masse (Kopplungsstärke der Gravitationswechselwirkung)

exp. Erfahrung: $m_s^{(i)} \sim m^{(i)}$ (alle Körper fallen gleich schnell)

bei Wahl der Gravitationskonstante $G = 6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg s}^2}$
 $m_s^{(i)} = m^{(i)}$

Einheiten: MKS (Meter, Kilogramm, Sekunde)

Größe	SI-Einheit	CGS-Einheit
Länge	m	cm
Zeit	s	s
Masse	kg	g
Geschw.	m s^{-1}	cm s^{-1}
Beschl.	m s^{-2}	$\text{Gal} = \text{cm s}^{-2}$ (Gal)
Kraft	$\text{N} = \text{kg m s}^{-2}$ (Newton)	$\text{dyn} = \text{g cm s}^{-2}$ (Dyn)
Energie	$\text{J} = \text{N m}$ (Joule)	$\text{erg} = \text{dyn cm}$ (Erg)
Leistung	$\text{W} = \text{J s}^{-1}$ (Watt)	erg s^{-1}

Schreibweise: $[\vec{v}] = \frac{\text{m}}{\text{s}}$ $[\vec{F}] = \text{kg} \frac{\text{m}}{\text{s}^2} = [m][\vec{v}]$

Hilfreich für Konsistenzcheck bei Rechnungen

0. Overview

- analytical mechanics: derive equations of motion using different frameworks of varying mathematical complexity
- deterministic systems: - determined by initial conditions
- causal: generated by forces

1. Newtonian mechanics

1.1 Approach

Kinematics & dynamics of systems of mass point without constraints

↑ ↑
 framework newton based
 using energy on forces
 instead of forces
 "geometry of motion"

1.2 Newton's laws of motion

I: A body remains at rest or in motion at a constant speed in a straight line, unless it is acted upon by a force

II: The net force on a body is equal (i) to the body's acceleration multiplied by its mass (ii) to the rate of change of its momentum.

III: If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

IV: Forces obey the superposition principle.

• event: $(\underline{r}(t), t) \in \mathbb{R}^3 \times \mathbb{R}$

• dynamic variable: $\underline{r}(t)$: trajectory

$$\underline{v}(t) = \frac{d}{dt} \underline{r}(t) = \underline{\dot{r}}(t) \text{ velocity}$$

• $I \Leftrightarrow$ system of inertia (Galilean transformation)

$$\underline{r}(t) = \underline{R} \underline{r}'(t') + \underline{v}_0 t + \underline{x}_0$$

$$t = t' + t_0, \quad \underline{R}: \text{rotational matrix}$$

• Newton's laws of motion are Galilean invariant

↳ existence of absolute space

↳ all inertial frames share a universal time

• II $\Leftrightarrow \underline{\vec{F}} = m \frac{d^2}{dt^2} \underline{r}(t)$ has a unique solution

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for initial conditions $(\underline{r}_0, t_0) \Rightarrow \underline{r}(t; \underline{r}_0, t_0)$

• III $\Leftrightarrow$ no external forces: overall momentum is conserved

• IV $\hookrightarrow$ force add like vectors

• mass : a measure of the body's inertia = gravitational mass

$$\underline{\vec{F}}^{(12)} = -G m^{(1)} m^{(2)} \frac{\underline{r}^{(1)} - \underline{r}^{(2)}}{|\underline{r}^{(1)} - \underline{r}^{(2)}|^3}, \quad G = 6.67 \cdot 10^{-11} \frac{\text{kg}^3}{\text{kg s}^2}$$

2. Das d'Alembertsche Prinzip

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2. D'Alembertsches Prinzip

1. Zwangsbedingungen und Zwangskräfte
2. Virtuelle Verrückungen
3. D'Alembertsches Prinzip der virtuellen Arbeit
4. Generalisierte Koordinaten
5. Lagrange-Gleichungen 2. Art

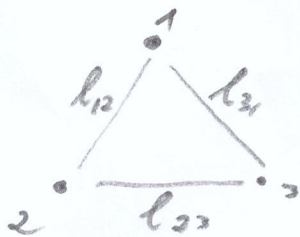
2.1 Zwangsbedingungen und Zwangskräfte

Ein System von N Massepunkten hat $f=3N$ **Freiheitsgrade**, wenn keine **Zwangsbedingungen** vorliegen.

Idee: Verminderung der Zahl der Freiheitsgrade durch

ii) **holonome** Zwangsbedingungen ("integrabel"/ganz gesetzt) $f_M(x_1, x_2, \dots, x_N) = 0$, $M=1, \dots, A$ (unabhängig: $\text{Rang}\left(\frac{\partial f_M}{\partial x_i}\right) = A$)

Bsp.: Starrer Körper aus 3 Teilchen



$$f_1 = |x_1 - x_2| - l_{12} = 0$$

$$f_2 = |x_2 - x_3| - l_{23} = 0$$

$$f_3 = |x_3 - x_1| - l_{31} = 0$$

Absstände fest

$$\Rightarrow f = 3N - A = 9 - 3 = 6$$

allgemein: Starrer Körper aus N Teilchen: $f_M = |x_i - x_j| - l_{ij} = 0$

N	anzahl Gleichungen	A	$f = 3N - A$
1	0	0	3
2	1	1	5
3	2	3	6
4	3	6	$4 \cdot 3 - 6 = 6$
5	3	9	$5 \cdot 3 - 9 = 6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N \geq 4$	3	$3N - 6$	6



(3x Translation, 3x Rotation)

Die Bahn $r_i(t)$ sollen 1 unabhängige Zwangsbedingungen

erfüllen: $\int_M (r_1(t), r_2(t), \dots, r_N(t)) dt = 0 \quad \forall t$

↳ Totale Zeitabhängigkeit: $\frac{dr_i}{dt}$

$$\frac{df_M}{dt} = \sum_{i=1}^N \underbrace{\nabla_{r_i} f_M}_{\text{Gradient}} \cdot \dot{r}_i + \frac{\partial f_M}{\partial t} = 0$$

↳ differentielle Schreibweise: (vollständiges Differential)

$$df_M = \sum_{i=1}^N \nabla_{r_i} f_M \cdot dr_i + \frac{\partial f_M}{\partial t} dt = 0$$

(ii) **nicht holonom** Zwangsbedingungen

$$\sum_{i=1}^N a_{Mi} (r_1, \dots, r_N, t) \cdot dr_i + a_{M0} (r_1, \dots, r_N, t) dt = 0 \quad M=1, \dots, A$$

Pfaffsche Differentialform (nicht integrabel, d.h. es existiert

kein integrierbarer Faktor g_M , so dass: $\sum g_M a_{Mi} \cdot dr_i + g_M a_{M0} dt = df_M$

also $g_M a_{Mi} = \nabla_{r_i} f_M$)

Bedeutung: beliebige Position $r_1, \dots, r_N$ zulässig, aber Bewegung lokal eingeschränkt. (etwa Geschwindigkeit längs der Bahn

$$\sum_{i=1}^N a_{Mi} \cdot \dot{r}_i + a_{M0} = 0)$$

Bsp: • Reagenium mit Auto auf Ebene: Jeder Punkt erreichbar, aber lokale Bewegung (mountain) davon lokal festgelegt gegeben.

• Schlittschuhlaufen: Kufen erlauben nur Vor & Zurück, aber nicht Seitwärts.

(iii) In beiden Fällen:

rheonom Zwangsbedingung (zeitabhängig)

skleonom ——— (nicht explizit zeitabhängig, statisch)

(iv) Ungleichungen (z.B. Gas in Behälter mit Wänden)

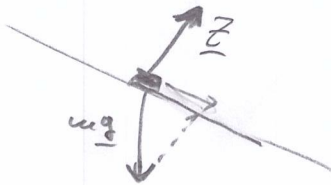
$\Rightarrow$ Bewegungsgleichungen: $m_i \ddot{\underline{r}}_i = \underbrace{\underline{F}_i}_{\text{äußere Kräfte}} + \underbrace{\sum_{j=1}^N \underline{F}_{ij}}_{\text{innere Kräfte}} =: \underline{X}_i \quad i=1, \dots, N$

müssen unter Nebenbedingungen $\left| \begin{array}{l} f_k(\underline{r}_1, \dots, \underline{r}_N, t) = 0 \\ \text{oder} \\ \sum_{i=1}^N \underline{a}_i \cdot \underline{r}_i + \underline{a}_0 = 0 \end{array} \right.$ gelöst werden.

Idee: Nebenbedingungen sollen durch **Zwangskräfte** $\underline{Z}_i$ erzwingen werden

$$m \ddot{\underline{r}}_i = \underline{X}_i + \underline{Z}_i \quad i=1, \dots, N$$

Bsp: 2D-Bewegung auf schiefer Ebene $\rightarrow$ effektive 1D-Bewegung



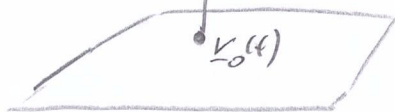
2.2 Virtuelle Verschiebungen

virtuelle Verschiebung $\{\delta \underline{r}_i\}$: infinitesimale Änderung der Koordinaten, die zu fester Zeit ($\delta t = 0$) die (holonom bzw. nichtholonom) Zwangsbedingungen erfüllen.

(Vgl.: reale Verschiebungen $d\underline{r}_i$ im Zeitintervall dt entlang der Bahn)

$$\delta f_k := \sum_{i=1}^N \frac{\partial f_k}{\partial \underline{r}_i} \cdot \delta \underline{r}_i = 0 \quad \text{bzw.} \quad \sum_{i=1}^N \underline{a}_i \cdot \delta \underline{r}_i = 0$$

Bsp: Bewegung in Ebene: $\underline{n} \cdot (\underline{r} - \underline{r}_0(t)) = 0$
 $\uparrow \underline{n}$: Normalenvektor Bewegung der Ebene



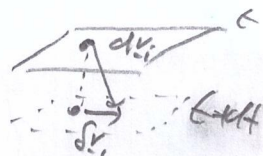
$\Rightarrow$ holonom Zwangsbedingung entspricht N Massenpunkten

$$f(\underline{r}_i) = \underline{n} \cdot (\underline{r}_i - \underline{r}_0(t)) = 0 \quad i=1, \dots, N$$

$$\Rightarrow df = \underline{n} \cdot (d\underline{r}_i - \underline{v}_0(t) dt) = 0 \quad \text{mit} \quad \underline{v}_0 = \frac{d}{dt} \underline{r}_0(t)$$

$$\text{also i.A.:} \quad \underline{n} \cdot d\underline{r}_i = \underline{n} \cdot \underline{v}_0(t) dt \neq 0$$

$$\text{also} \quad df = \underline{n} \cdot \delta \underline{r}_i = 0 \Rightarrow \delta \underline{r}_i \perp \underline{n} \quad \text{bei festgehaltener Ebene}$$



23 D'Alambertsches Prinzip der virtuellen Arbeit

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System von N Massenpunkten mit Zwangsbedingungen:

↳ Bewegungsgleichungen mit Zwangskräften:

$$m_i \ddot{\underline{x}}_i - \underline{X}_i = \underline{Z}_i \quad i=1, \dots, N$$

$$\Rightarrow \sum_{i=1}^N (m_i \ddot{\underline{x}}_i - \underline{X}_i) \cdot \delta \underline{x}_i = \sum_{i=1}^N \underline{Z}_i \cdot \delta \underline{x}_i$$

virtuelle Arbeit der eige Kräfte & der Zwangskräfte

Kräfte

Bsp: (i) Bewegung auf Fläche $f(\underline{x}, t) = 0$ (ebene Ebene $\underline{x} \cdot (\underline{x}_i - \underline{x}_0(t)) = 0$)

↳ Zwangskräfte $\perp$ Fläche $\Rightarrow \underline{Z}_i = \lambda_i (\underline{x}_i, \dots, \underline{x}_N, t) \nabla f$
 $\underline{\underline{x}}_i$
 $= n$ für Ebene

virtuelle Arbeit: $\underline{Z}_i \cdot \delta \underline{x}_i = \lambda_i \nabla f \cdot \delta \underline{x}_i$ ($= \lambda_i \delta f$)
 $\underline{\underline{x}}_i$
 $\perp$ Fläche
 $\perp$ Fläche

$$\Rightarrow \sum_{i=1}^N \underline{Z}_i \cdot \delta \underline{x}_i = 0 \quad \text{virtuelle Arbeit verschwindet}$$

oder reelle Arbeit: $\sum_{i=1}^N \underline{Z}_i \cdot d\underline{x}_i \neq 0$ im Allg. Fall

(ii) starre Körper $f_{ij} = |\underline{x}_i - \underline{x}_j| - l_{ij} = 0$

↳ Zwangskräfte in Richtung $\underline{x}_i - \underline{x}_j \Rightarrow \underline{Z}_{ij} = \lambda_{ij} \frac{\underline{x}_i - \underline{x}_j}{|\underline{x}_i - \underline{x}_j|} = \lambda_{ij} \frac{\underline{x}_i - \underline{x}_j}{r_{ij}}$

actio = reactio: $\underline{Z}_{ij} = -\underline{Z}_{ji} \Rightarrow \lambda_{ij} = \lambda_{ji}$

$\Rightarrow$ gesamte Zwangskraft auf m_i : $\underline{Z}_i = \sum_{j \neq i} \underline{Z}_{ij} = \sum_{j \neq i} \lambda_{ij} \frac{\underline{x}_i - \underline{x}_j}{r_{ij}}$

$$\Rightarrow \underline{Z}_i \cdot \delta \underline{x}_i = \sum_{j \neq i} \lambda_{ij} \frac{\underline{x}_i - \underline{x}_j}{r_{ij}} \cdot \delta \underline{x}_i \neq 0 \quad \text{i. d. R.}$$

↳ verschwindet nicht für einzelne Körper, aber:

$$\sum_{i=1}^N \underline{Z}_i \cdot \delta \underline{x}_i = \sum_{ij} \lambda_{ij} \frac{\underline{x}_i - \underline{x}_j}{r_{ij}} \cdot \delta \underline{x}_i$$

$$= \frac{1}{2} \sum_{ij} \lambda_{ij} \frac{\underline{x}_i - \underline{x}_j}{r_{ij}} \delta (\underline{x}_i - \underline{x}_j) = \frac{1}{2} \sum_{ij} \lambda_{ij} \delta r_{ij} \stackrel{\text{Abstandsfunktion}}{=} 0$$

$$= \delta r_{ij}, \text{ da } \delta |\underline{r}| = \delta \left(\sqrt{(\underline{r} \cdot \underline{r})} \right) = \frac{1}{2} (\underline{r} \cdot \underline{r})^{-\frac{1}{2}} 2 \underline{r} \cdot \delta \underline{r} = \frac{\underline{r}}{r} \cdot \delta \underline{r}$$

Allgemeine Forderung: $\sum_{i=1}^N \ddot{x}_i \delta x_i = 0$

$\Rightarrow \sum_{i=1}^N (m_i \ddot{x}_i - K_i) \cdot \delta x_i = 0$ d'Alembert'sches Prinzip

• Differenzialprinzip / Variationsprinzip

Der wirklichen angenommen Zustand ist ein Extremalzustand (in dem Sinn, dass die gesamte virtuelle Arbeit Null ist.)

§ Variationsprinzip mit Nebenbedingungen:

(allg. Notation: $\ddot{x}_j \rightarrow \ddot{x}_j$, $K_j \rightarrow K_j$, $\phi_j^\mu \rightarrow \phi_j^\mu$ Koordinaten $i=1, \dots, N, j=1, \dots, 3N$)

$$\sum_{j=1}^{3N} (m_j \ddot{x}_j - K_j) \delta x_j = 0 \quad \text{Nebenbedingungen} \quad \sum_{j=1}^{3N} \phi_j^\mu \delta x_j = 0, \mu=1, \dots, 1$$

Lösung mit der Methode der **Lagrange-Multiplikation**

Falls δx_i frei wählbar, dann würde $(m_j \ddot{x}_j - K_j) = 0$ gelten

Idee: Umschreiben, sodass ein Satz von Faktoren frei variierbar ist

(i) Addieren Nebenbedingungen mit Lagrange-Multiplikation λ_μ :

$$\sum_{j=1}^{3N} \left(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^1 \lambda_\mu \phi_j^\mu \right) \delta x_j = 0$$

(ii) Eliminieren $\delta x_1, \dots, \delta x_1$ aus den Nebenbedingungen, dann sind

$\delta x_{1+1}, \dots, \delta x_N$ frei variierbar.

(iii) Bestimme $\lambda_1, \dots, \lambda_1$ sodass

$$m_j \ddot{x}_j - K_j - \sum_{\mu=1}^1 \lambda_\mu \phi_j^\mu = 0 \quad \text{für } j=1, \dots, 1$$

(λ_μ für $\lambda_\mu^{(1)}$ als Funktionen von $\ddot{x}_j^{(1)}$)

$$\Rightarrow \sum_{j=1+1}^{3N} \left(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^1 \lambda_\mu^{(1)} \phi_j^\mu \right) \delta x_j = 0$$

frei variierbar

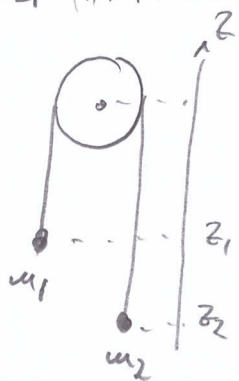
$$\Rightarrow m_j \ddot{x}_j - K_j - \sum_{\mu=1}^1 \lambda_\mu^{(1)} \phi_j^\mu = 0 \quad \text{Lagrange-Gleichungen 1. Art}$$

"Zwangsbedingung"

Für holonom Zwangsbedingungen $m_j \ddot{x}_j - K_j - \sum_{\mu=1}^1 \lambda_\mu \frac{\partial \phi_\mu}{\partial x_j} = 0$

Bsp: Atwoodsche Fallmaschine

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$$(m_1 \ddot{z}_1 - X_1) \delta z_1 + (m_2 \ddot{z}_2 - X_2) \delta z_2 = 0$$

$\uparrow$ $\uparrow$
 $-m_1 g$ $-m_2 g$

$$z_1 + z_2 = \text{const.} \Rightarrow \ddot{z}_1 = -\ddot{z}_2$$

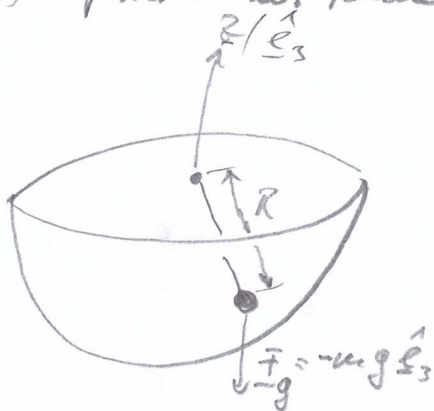
(Zwangsgl.)

$$\delta z_1 = -\delta z_2$$

$$\Rightarrow \underbrace{[(m_1 + m_2) \ddot{z}_1 + (m_1 - m_2) g]}_{=0} \delta z = 0 \quad \text{für bel. } \delta z$$

$$\ddot{z}_1 = -\frac{m_1 - m_2}{m_1 + m_2} g$$

(ii) Sphärisches Pendel



$$\text{Zwangsbedg: } f(r(t)) = r(t)^2 - R^2 = 0$$

$$\text{Trick: } \dot{f}(r) = 2r \cdot \dot{r} = 0$$

$$\dot{f}''(r) = 2\dot{r}^2 + 2r \cdot \ddot{r} = 0$$

$$\text{Bewegungsgl: } m\ddot{r} = m\underline{g} + \underline{Z} \stackrel{\underline{r}}{\perp} = m\underline{g} + 2\lambda \underline{r}$$

$$\underline{Z}: \text{in Richtung } \underline{r} \text{ plausibel} \Rightarrow \text{Annahme: } \underline{Z} = 2\lambda \underline{r}$$

$$\Rightarrow m\underline{r} \cdot \ddot{r} = m\underline{r} \cdot \underline{g} + 2\lambda \underbrace{\underline{r} \cdot \underline{r}}_{=R}$$

$$-m\dot{r}^2 = m\underline{r} \cdot \underline{g} + 2\lambda R^2 \Rightarrow 2\lambda = -\frac{m}{R^2} (\dot{r}^2 + \underline{r} \cdot \underline{g})$$

$$\Rightarrow \underline{Z} = -\frac{m}{R^2} (\dot{r}^2 + \underline{r} \cdot \underline{g}) \underline{r} \quad (\text{Geschwindigkeit hat Abhängigkeit!})$$

$\uparrow$
 Zentrifugal- & Coriolis
 Kraft

$$\Rightarrow \ddot{r} = \underline{g} - \frac{1}{R^2} (\dot{r}^2 + \underline{r} \cdot \underline{g}) \underline{r}$$

$$(\text{Allgemein: } \underline{Z} = \lambda \underline{\nabla} f \Rightarrow \underline{Z} = \lambda 2\underline{r})$$

Gradient $\perp$ auf Fläche $f=0$, wie auch $\underline{Z}$, wenn Tangentialbew.

2 D'Alembert's principle

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2.1 Constraints and forces

System of N mass points $\rightarrow 3N$ degrees of freedom (without constraints)

idea: reduction of degrees of freedom by constraints.

(i) holonomic: $f_\mu(\underline{r}_1, \dots, \underline{r}_N, t) = 0, \mu = 1, \dots, A \Rightarrow f = 3N - A$

$$\text{total differential: } df_\mu = \sum_{i=1}^N \nabla_{\underline{r}_i} f_\mu \cdot d\underline{r}_i + \frac{\partial f_\mu}{\partial t} dt = 0$$

(ii) nonholonomic: $\sum_{i=1}^N \underline{a}_{\mu i}(\underline{r}_1, \dots, \underline{r}_N, t) \cdot d\underline{r}_i + \underline{a}_{\mu 0}(\underline{r}_1, \dots, \underline{r}_N, t) dt = 0$

but cannot be written as $df_\mu = \dots = 0$ (even with an

integrating factor: $\exists g_\mu: \underline{a}_{\mu i} = g_\mu \nabla_{\underline{r}_i} f_\mu$

(mean: motion locally restricted)

(a) rheonomic: explicitly time dependent

(b) scleronomous: not explicitly time dependent

$$\text{equations of motion: } m_i \ddot{\underline{r}}_i = \underbrace{\underline{F}_i}_{\text{external}} + \underbrace{\sum_{j=1}^N \underline{F}_{ij}}_{\text{internal forces}} =: \underline{X}_i, \quad i=1, \dots, N$$

$\hookrightarrow$ solution given constraints.

Idea: turn constraints into forces governing the motion:

$$m_i \ddot{\underline{r}}_i = \underline{X}_i + \underline{Z}_i$$

2.2 Virtual displacements

• (hypothetical) deviation from trajectory obeying the constraints (for time t_{fix})

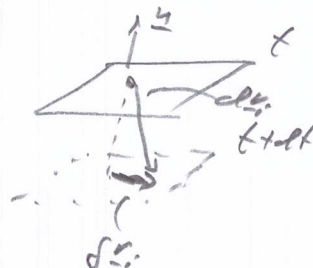
• ex. problem on oscillating plane: $f = \underline{n} \cdot (\underline{r} - \underline{r}_0(t)) = 0$
position of plane

$$\Rightarrow df = \underline{n} \cdot (d\underline{r}_i - \underline{v}_0(t) dt) = 0 \quad \text{with } \underline{v}_0 = \frac{d}{dt} \underline{r}_0(t)$$

$$\Rightarrow \underline{n} \cdot d\underline{r}_i = \underline{v}_0(t) dt \quad (\text{work of } \underline{f} \text{ not zero})$$

$$\text{but } \delta f = \underline{n} \cdot \delta \underline{r}_i \stackrel{!}{=} 0 \quad \Rightarrow \delta \underline{r}_i \perp \underline{n}$$

$\uparrow$
fixing time



2.3 D'Alembert's principle

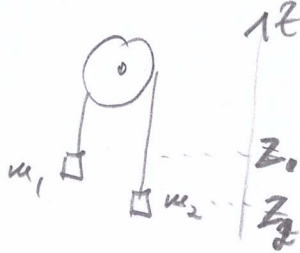
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virtual work: $\sum_{i=1}^N \vec{z}_i \cdot d\vec{r}_i = 0 \Rightarrow$ D'Alembert's principle: $\sum_{i=1}^N (m_i \ddot{\vec{r}}_i - \vec{X}_i) \cdot d\vec{r}_i = 0$

ex: (i) $\sum_{i=1}^N \vec{z}_i \cdot d\vec{r}_i = \sum_{i=1}^N \underbrace{\vec{p}_i}_{=\vec{p}} \cdot \underbrace{d\vec{r}_i}_{\parallel \text{ to plane}} = 0$

(notes: no constraints explicitly included)

(ii) Atwood machine: m_1 & m_2 connected via a pulley by a string



$z_1 + z_2 = \text{const} \Rightarrow \ddot{z}_1 = -\ddot{z}_2, \delta z_1 = -\delta z_2$

$\left(m_1 \ddot{z}_1 - \underbrace{(-m_1 g)}_{X_1} \right) \delta z_1 + \left(m_2 \ddot{z}_2 - \underbrace{(-m_2 g)}_{X_2} \right) \delta z_2 = 0$

$\Rightarrow \underbrace{\left[(m_1 + m_2) \ddot{z}_1 + (m_1 - m_2) g \right]}_{\stackrel{!}{=} 0} \delta z_1 = 0 \text{ for arbitrary } \delta z_1$

$\Rightarrow \ddot{z}_1 = - \frac{m_1 - m_2}{m_1 + m_2} g$

general notation: $\vec{r}_i \rightarrow x_j, \vec{X}_i \rightarrow K_j, \frac{d}{dt} \vec{r}_i \rightarrow \dot{\phi}_j^K$

$\Rightarrow \sum_{j=1}^{3N} \left(m_j \ddot{x}_j - K_j \right) \delta x_j = 0 \text{ given } \sum_{j=1}^{3N} \dot{\phi}_j^K \delta x_j = 0$

idea: Lagrange multipliers (add constraints to equations of motion)

$\hookrightarrow \sum_{j=1}^{3N} \left(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_{\mu} \dot{\phi}_j^{\mu} \right) \delta x_j = 0$

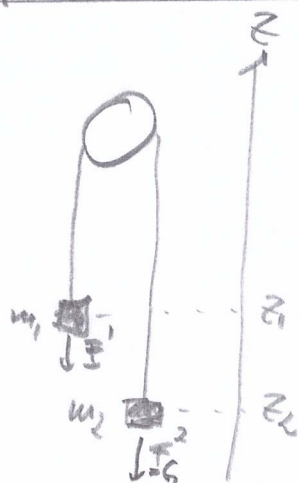
choose/determine $\lambda_{(1)}, \dots, \lambda_{(\Lambda)}$ such that $m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_{\mu} \dot{\phi}_j^{\mu} = 0, j=1, \dots, 3N$

insert into remaining equations: $\underbrace{\sum_{j=1}^{3N} \left(m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_{\mu}^{(1)} \dot{\phi}_j^{\mu} \right) \delta x_j}_{=0} = 0 \text{ free}$

$\Rightarrow$ Lagrange's equations of the first kind:

$m_j \ddot{x}_j - K_j - \sum_{\mu=1}^{\Lambda} \lambda_{\mu}^{(1)} \dot{\phi}_j^{\mu} = 0 \quad j=1, \dots, 3N$

Übenschub Beispiel Atwoodsche Fallmaschine auf 4 Variablen (Beispiel)



(i) Newton: $\vec{T} = m \vec{g}$

$$\vec{T}^{(1)} = -m_1 \vec{g} + m_2 \vec{g} = -(m_1 - m_2) \vec{g}$$

$$\vec{T}_{\text{Ges}} = m_{\text{ges}} \ddot{\vec{z}} = (m_1 + m_2) \ddot{\vec{z}}$$

$$\Rightarrow (m_1 + m_2) \ddot{\vec{z}} = -(m_1 - m_2) \vec{g}$$

$$\Rightarrow \ddot{\vec{z}} = - \frac{m_1 - m_2}{m_1 + m_2} \vec{g}$$

(ii) D'Alembert $\sum_{i=1}^N \vec{z}_i \cdot \delta \vec{r}_i = 0 = \sum_{i=1}^N (m_i \ddot{\vec{r}}_i - \vec{F}_i) \cdot \delta \vec{r}_i$

$$\delta z_1 = -\delta z_2, \quad \vec{z}_1 = \vec{z}_2 \Rightarrow \sum_{i=1}^2 \vec{z}_i \cdot \delta \vec{r}_i = -\vec{z} (\delta z_1 + \delta z_2) = \vec{z} (\delta z_1 - \delta z_2) = 0$$

$$(m_1 \ddot{\vec{z}}_1 - (-m_1 \vec{g})) \delta \vec{z}_1 + (m_2 \ddot{\vec{z}}_2 - (-m_2 \vec{g})) \delta \vec{z}_2 = 0$$

$\uparrow$
 $= -\ddot{\vec{z}}_1$
 $\quad \quad \quad$
 $\quad \quad \quad$
 $\downarrow$
 $= -\delta \vec{z}_1$

$$\Rightarrow \left[(m_1 + m_2) \ddot{\vec{z}}_1 + (m_1 - m_2) \vec{g} \right] \delta \vec{z}_1 = 0$$

$$\stackrel{!}{=} 0 \Rightarrow \ddot{\vec{z}}_1 = - \frac{m_1 - m_2}{m_1 + m_2} \vec{g}$$

(iii) Lagrange I: $\dot{r}_1 = \dot{r}_2 = \dot{r}_1 = \dot{r}_2 = 0, \quad z_1 + z_2 = l = \text{const}$

$$m_j \ddot{x}_j - \vec{F}_j - \sum_{\mu=1}^1 \lambda_{\mu} \frac{\partial f_{\mu}}{\partial x_j} = 0 \quad \Rightarrow f = z_1 + z_2 - l = 0$$

$$\Rightarrow df = dz_1 + dz_2 = 0$$

$$\Rightarrow \text{I: } m_1 \ddot{z}_1 - (-m_1 \vec{g}) - \lambda \cdot 1 = 0$$

$$\text{und } \dot{z}_1 + \dot{z}_2 = 0, \quad \ddot{z}_1 = -\ddot{z}_2$$

$$\text{II: } m_2 \ddot{z}_2 - (m_2 \vec{g}) - \lambda \cdot 1 = 0$$

$$\text{I+II: } m_1 \ddot{z}_1 + m_2 \ddot{z}_2 + (m_1 + m_2) \vec{g} - 2\lambda = 0$$

$$\Leftrightarrow (m_1 - m_2) \ddot{z}_1 + (m_1 + m_2) \vec{g} = 2\lambda$$

$$\text{I-II: } m_1 \ddot{z}_1 - m_2 \ddot{z}_2 + (m_1 - m_2) \vec{g} = 0$$

$$\Leftrightarrow \ddot{z}_1 = - \frac{m_1 - m_2}{m_1 + m_2} \vec{g}$$

$$\Leftrightarrow \frac{(m_1 - m_2)^2}{m_1 + m_2} \vec{g} + (m_1 + m_2) \vec{g} = 2\lambda \Rightarrow \lambda = \frac{g}{2} \frac{(m_1 + m_2)^2 - (m_1 - m_2)^2}{m_1 + m_2} = 2g \frac{m_1 m_2}{m_1 + m_2}$$

(siehe auch 1))

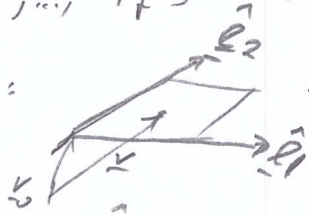
2.4 Generalisierte Koordinaten

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Idee: Finde $f = 3N - 1$ Koordinaten, die unabhängig voneinander sind und dennoch die Zwangsbedingungen erfüllen. *Generalisierte*

Koordinaten $\{q_1, \dots, q_f\}$ $f = \#$ Freiheitsgrade (wie bei holonom ZB)

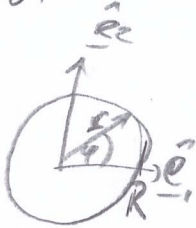
Bsp: (i) Ebene:



$$\underline{r}(t) = \underline{r}_0(t) + q_1 \underline{\hat{e}}_1 + q_2 \underline{\hat{e}}_2 \Rightarrow f=2$$

mit benutzte Koordinatengitter

(ii) Kreis:



$$\underline{r}(t) = R (\cos \varphi \underline{\hat{e}}_1 + \sin \varphi \underline{\hat{e}}_2) \Rightarrow f=1$$

Ziel: Virtuelle Verschiebungen $\delta \underline{r}_i$ durch $\delta q_1, \dots, \delta q_f$ ausdrücken:

$$\underline{r}_i(q_1, \dots, q_f, t) \quad \delta \underline{r}_i = \sum_{j=1}^f \left(\frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j \Rightarrow \sum_{j=1}^f \underline{x}_i \cdot \delta \underline{r}_i = \sum_{j=1}^f \left(\sum_{i=1}^N \underline{x}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j = 0$$

generalisierte Kräfte

2.5 Lagrange-Gleichungen 2. Art

$$\text{d'Alembertsches Prinzip: } \sum_{i=1}^N m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{i=1}^N \underline{x}_i \cdot \underline{f}_i = \sum_{j=1}^f Q_j \delta q_j$$

• links steht durch Edwergleichungen durch Definition der *Kinetischen*

$$\text{Energie } T = \sum_{i=1}^N \frac{1}{2} m_i v_i^2 \quad \text{zu: } \sum_{i=1}^N m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{j=1}^f \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

$$\Rightarrow \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \right] \quad \text{mit } j=1, \dots, f$$

Lagrange-Gleichungen 2. Art

- gilt nur für holonom Zwangsbedingungen (sonst kein generalisierte Koordinaten)

• Für *konservative Kräfte*: $Q_j = -\frac{\partial V}{\partial q_j}$ mit $V(q_1, \dots, q_f, t)$

$$\Rightarrow \frac{d}{dt} \frac{\partial T-V}{\partial \dot{q}_j} - \frac{\partial T-V}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Lagrange-Funktion

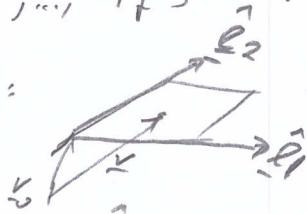
2.4 Generalisierte Koordinaten

12

Idee: Finde $f = 3N - 1$ Koordinaten, die unabhängig voneinander sind und dennoch die Zwangsbedingungen erfüllen. *Generalisierte*

Koordinaten $\{q_1, \dots, q_f\}$ $f = \#$ Freiheitsgrade (wie bei holonom ZB)

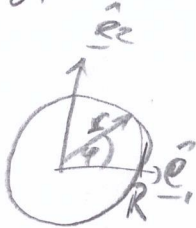
Bsp: (i) Ebene:



$$\underline{r}(t) = \underline{r}_0(t) + q_1 \underline{\hat{e}}_1 + q_2 \underline{\hat{e}}_2 \Rightarrow f=2$$

unabhängige Koordinatengrößen

(ii) Kreis:



$$\underline{r}(t) = R (\cos \varphi \underline{\hat{e}}_1 + \sin \varphi \underline{\hat{e}}_2) \Rightarrow f=1$$

Ziel: Virtuelle Verschiebungen $\delta \underline{r}_i$ durch $\delta q_1, \dots, \delta q_f$ ausdrücken:

$$\underline{r}_i(q_1, \dots, q_f, t) \quad \delta \underline{r}_i = \sum_{j=1}^f \left(\frac{\partial \underline{r}_i}{\partial q_j} \right) \delta q_j \Rightarrow \sum_{i=1}^N \underline{x}_i \cdot \delta \underline{r}_i = \sum_{j=1}^f \underbrace{\left(\sum_{i=1}^N \underline{x}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \right)}_{Q_j} \delta q_j = 0$$

generalisierte Kräfte

2.5 Lagrange-Gleichungen 2. Art

$$\text{D'Alembertsches Prinzip: } \sum_{i=1}^N m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{i=1}^N \underline{x}_i \cdot \underline{f}_i = \sum_{j=1}^f Q_j \delta q_j$$

Kann sich durch Lösungsumformungen durch Definition der *Kinetischen*

$$\text{Energie } T = \sum_{i=1}^N \frac{1}{2} m_i v_i^2 \quad \text{zu: } \sum_{i=1}^N m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{j=1}^f \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad \text{mit } j=1, \dots, f$$

Lagrange-Gleichungen 2. Art

- gilt nur für holonom Zwangsbedingungen (sonst kein generalisierte Koordinaten)

• Für *konervative Kräfte*: $Q_j = -\frac{\partial V}{\partial q_j}$ mit $V(q_1, \dots, q_f, t)$

$$\Rightarrow \frac{d}{dt} \frac{\partial T-V}{\partial \dot{q}_j} - \frac{\partial T-V}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Lagrange-Funktion

2.4 Generalized coordinates

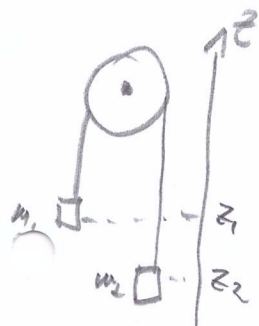
12a

set $\{q_1, \dots, q_f\}$ of independent coordinates that obey given constraints

idea: express virtual displacements $\delta \underline{r}_i$ ($i=1, \dots, N$) by δq_k ($k=1, \dots, f$)
 δx_j ($j=1, \dots, 3N$)

with $f = |\text{degrees of freedom}|$.

case study Atwood's machine



(i) Newton $\underline{F} = m \underline{a}$

(ii) D'Alembert $\sum_{i=1}^N \underline{z}_i \cdot \delta \underline{r}_i = 0 = \sum_{i=1}^N (m_i \ddot{\underline{r}}_i - \underline{X}_i) \cdot \delta \underline{r}_i$

(iii) Lagrange I $m_j \ddot{x}_j - K_j - \sum_{\mu=1}^f \lambda_{\mu} \frac{\partial f_{\mu}}{\partial x_j} = 0$

$$\ddot{z}_1 = - \frac{m_1 - m_2}{m_1 + m_2} g$$

$$(iv) \text{ Lagrange II: } \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0, \quad j=1, \dots, f$$

$$\text{hier } f=1 \Rightarrow \text{weil } z_1 + z_2 - l = \text{const} \Rightarrow z_1 = q, \quad z_2 = l - q$$

$$T = \frac{1}{2} m_1 \dot{z}_1^2 + m_2 \dot{z}_2^2 = \frac{1}{2} (m_1 + m_2) \dot{q}^2$$

$$V = m_1 g z_1 + m_2 g z_2 = g (m_1 q + m_2 (l - q)) = g (m_1 - m_2) q + m g l$$

$$\Rightarrow L = \frac{1}{2} (m_1 + m_2) \dot{q}^2 - g (m_1 - m_2) q - m g l$$

$$\Rightarrow \frac{d}{dt} [(m_1 + m_2) \dot{q}] + g (m_1 - m_2) = 0$$

$$\Rightarrow (m_1 + m_2) \ddot{q} + g (m_1 - m_2) = 0$$

$$\Rightarrow \ddot{q} = - \frac{m_1 - m_2}{m_1 + m_2} g$$

$$(iv) \text{ Lagrange II: } \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j=1, \dots, f$$

$$\text{hier } f=1 \Rightarrow \text{weil } z_1 + z_2 - l = \text{const} \Rightarrow z_1 = q, \quad z_2 = l - q$$

$$T = \frac{1}{2} m_1 \dot{z}_1^2 + m_2 \dot{z}_2^2 = \frac{1}{2} (m_1 + m_2) \dot{q}^2$$

$$V = m_1 g z_1 + m_2 g z_2 = g (m_1 q + m_2 (l - q)) = g (m_1 - m_2) q + m g l$$

$$\Rightarrow L = \frac{1}{2} (m_1 + m_2) \dot{q}^2 - g (m_1 - m_2) q - m g l$$

$$\Rightarrow \frac{d}{dt} [(m_1 + m_2) \dot{q}] + g (m_1 - m_2) = 0$$

$$\Rightarrow (m_1 + m_2) \ddot{q} + g (m_1 - m_2) = 0$$

$$\Rightarrow \ddot{q} = - \frac{m_1 - m_2}{m_1 + m_2} g$$

Einschub: Berechnung der linken Seite

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$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^f \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) \delta \mathbf{q}_j$$

$$= \sum_{j=1}^f \sum_{i=1}^N \left\{ \frac{d}{dt} m_i \dot{\mathbf{r}}_i \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) - m_i \dot{\mathbf{r}}_i \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) \right\} \delta \mathbf{q}_j$$

Mit $\frac{d}{dt} \mathbf{r}_i = \dot{\mathbf{r}}_i = \sum_{j=1}^f \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) \dot{\mathbf{q}}_j + \frac{\partial \mathbf{r}_i}{\partial t} = \dot{\mathbf{r}}_i$ folgt $\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) = \frac{\partial}{\partial \mathbf{q}_j} \dot{\mathbf{r}}_i$, und $\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\mathbf{q}}_j} = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j}$

damit: $\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) = \sum_k \left(\frac{\partial^2 \mathbf{r}_i}{\partial \mathbf{q}_k \partial \mathbf{q}_j} \right) \dot{\mathbf{q}}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial \mathbf{q}_j}$

$$\frac{\partial}{\partial \mathbf{q}_j} \dot{\mathbf{r}}_i = \frac{\partial}{\partial \mathbf{q}_j} \left\{ \sum_k \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_k} \right) \dot{\mathbf{q}}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right\} = \sum_k \left(\frac{\partial^2 \mathbf{r}_i}{\partial \mathbf{q}_j \partial \mathbf{q}_k} \right) \dot{\mathbf{q}}_k + \frac{\partial^2 \mathbf{r}_i}{\partial \mathbf{q}_j \partial t}$$

$$\Rightarrow \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_j \sum_i \left\{ \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) - m_i \dot{\mathbf{r}}_i \left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \mathbf{q}_j} \right) \right\} \delta \mathbf{q}_j \quad (*)$$

$= \frac{\partial}{\partial \mathbf{q}_j} \dot{\mathbf{r}}_i$

mit $m_i \dot{\mathbf{r}}_i \left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) = \frac{\partial}{\partial \dot{\mathbf{q}}_j} \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right)$ und $m_i \dot{\mathbf{r}}_i \left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \mathbf{q}_j} \right) = \frac{\partial}{\partial \mathbf{q}_j} \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right)$ folgt:

$$(*)^2 \sum_{j=1}^f \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}_j} - \frac{\partial T}{\partial \mathbf{q}_j} - Q_j \right\} \delta \mathbf{q}_j = 0, \quad T = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$$

$\uparrow$
generalisierter Impuls

$$\Rightarrow \boxed{\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial T}{\partial \mathbf{q}_j} = Q_j}$$

Für konservative Kräfte: $Q_j = - \frac{\partial V}{\partial q_j}$

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$$(V(r_1(q_1, \dots, q_f, t), r_2(q_1, \dots, q_f, t), \dots, r_N(q_1, \dots, q_f, t))) = V(q_1, \dots, q_f, t)$$

$$\Rightarrow \frac{\partial V}{\partial q_j} = 0 \quad \text{im allgemeinen (s.o.)}$$

und die Definition der Lagrange-Funktion $L(\{q_j\}_{j=1}^f, \{\dot{q}_j\}_{j=1}^f, t) = T - V$ folgt:

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad j=1, \dots, f}$$

Anwandschema / Kochrezept:

Kochrezept der Lagrange-Gleichungen 2. Art:

1. Auswählen generalisierter Koordinaten: $q_1, \dots, q_f$, die die (holonomen) Zwangsbedingungen erfüllen: $\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_f, t) \equiv \mathbf{r}_i(q_k, t)$
2. Berechnen von Geschwindigkeiten $\mathbf{v}_i(q_k, \dot{q}_k, t) = \frac{d}{dt} \mathbf{r}_i(q_k, t)$ und kinetischer Energie $T(q_k, \dot{q}_k, t) = \frac{1}{2} \sum_{i=1}^N m_i \mathbf{v}_i(q_k, \dot{q}_k, t)^2$
3. Potenzielle Energie für konservative Kräfte: $V(q_k, t) = V(\mathbf{r}_i(q_k, t))$
 $\Rightarrow$ Lagrange-Funktion $L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k, t) - V(q_k, t)$
(Potenzielle/Lage-) Energie für nichtkonservative Kräfte: $Q_j = \sum_{i=1}^N \mathbf{X}_i \frac{\partial}{\partial q_j} \mathbf{r}_i(q_1, \dots, q_f, t)$
4. Aufstellung der Bewegungsgleichungen ($j = 1, \dots, f$):

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{bzw.} \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

5. Lösen der Bewegungsgleichungen

Bem.: Bei Geschwindigkeitsabhängigen Potenzial

- etwa ein elektromagnetisches Potenzial: $V_{em} = e(\phi - \mathbf{v} \cdot \mathbf{A})$ -

ergibt sich: $Q_j = - \frac{\partial V}{\partial q_j} + \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_j}$

Idee: $\sum_{i=1}^N \mathbf{X}_i \cdot \mathbf{F}_i = \sum_{i=1}^N \left(- \frac{\partial V}{\partial \mathbf{r}_i} + \frac{d}{dt} \frac{\partial V}{\partial \dot{\mathbf{r}}_i} \right) \mathbf{F}_i$

Für V_{em} : $Q_j = - \frac{\partial V_{em}}{\partial x_j} + \frac{d}{dt} \frac{\partial V_{em}}{\partial \dot{x}_j} = e \left(- \frac{\partial \phi}{\partial x_j} + \dot{\mathbf{r}} \cdot \frac{\partial \mathbf{A}}{\partial x_j} - \frac{dA_j}{dt} \right)$

$$= e \left(- \frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t} - \sum_{k=1}^3 \frac{\partial A_k}{\partial x_k} \dot{x}_k + \sum_{k=1}^3 \dot{x}_k \frac{\partial A_k}{\partial x_j} \right)$$

$$= e \left(- \frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t} + \sum_{k=1}^3 \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \right) \dot{x}_k \right) \Rightarrow \mathbf{F}_{el} = e \left(- \nabla \phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \right)$$

$$\text{Allgemein: } \sum_{i=1}^N \underline{x}_i \cdot \underline{\dot{r}}_i = \sum_{j=1}^f \left(\sum_{i=1}^N \underline{x}_i \cdot \frac{\partial \underline{r}_i}{\partial \underline{q}_j} \right) \delta \underline{q}_j$$

$$= \sum_{j=1}^f \left(\sum_{i=1}^N \left(-\frac{\partial V}{\partial \underline{r}_i} + \frac{d}{dt} \frac{\partial V}{\partial \dot{\underline{r}}_i} \right) \cdot \frac{\partial \underline{r}_i}{\partial \underline{q}_j} \right) \delta \underline{q}_j$$

$$= \sum_{j=1}^f \left(-\frac{\partial V}{\partial \underline{q}_j} + \sum_{i=1}^N \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{\underline{r}}_i} \right) \cdot \frac{\partial \underline{r}_i}{\partial \underline{q}_j} \right) \delta \underline{q}_j$$

$$\frac{\partial V}{\partial \underline{q}_j} = \sum_{i=1}^N \frac{\partial V}{\partial \underline{r}_i} \frac{\partial \underline{r}_i}{\partial \underline{q}_j} = \sum_{i=1}^N \frac{\partial V}{\partial \dot{\underline{r}}_i} \frac{\partial \dot{\underline{r}}_i}{\partial \underline{q}_j}$$

$$= \sum_{j=1}^f \left(-\frac{\partial V}{\partial \underline{q}_j} + \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{\underline{q}}_j} \right) \right) \delta \underline{q}_j$$

$$= \sum_{j=1}^f \underbrace{\left(-\frac{\partial V}{\partial \underline{q}_j} + \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{\underline{q}}_j} \right) \right)}_{Q_j} \delta \underline{q}_j$$

$$(i) \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = \sum_{i=1}^N m \ddot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \frac{d}{dt} \left(\sum_{i=1}^N \frac{m_i}{2} \dot{\vec{x}}_i^2 \right) = \frac{d}{dt} T$$

$$(ii) \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{\partial V_{ij}}{\partial \vec{r}_i} \right) + \frac{\partial V_i}{\partial \vec{r}_i} \right) \cdot \dot{\vec{r}}_i$$

Konservative Kräfte

$$= - \frac{d}{dt} \left(\frac{1}{2} \sum_{i,j=1}^N V_{ij}(\vec{r}_{ij}) + \sum_{i=1}^N V_i(\vec{r}_i) \right)$$

$$= - \frac{d}{dt} V$$

$$\left(\frac{d}{dt} V_{ij}(\vec{r}_{ij}) = \left(\frac{\partial V_{ij}}{\partial \vec{r}_{ij}} \right) \cdot \dot{\vec{r}}_i + \left(\frac{\partial V_{ij}}{\partial \vec{r}_{ij}} \right) \cdot \dot{\vec{r}}_j \right)$$

$$i+ii \Rightarrow \frac{dT}{dt} + \frac{dV}{dt} = 0$$

(evtl. können dissipative Prozesse, z.B. Reibung, die Energieerhaltung stören/aufheben)

$$\frac{d}{dt}(T+V) = \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i$$

Wird bei Berücksichtigung von Zwangsbedingungen: $\sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i$ auf null reduziert

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$$\frac{d}{dt}(T+V) = - \sum_{\mu=1}^M \lambda_{\mu} \frac{\partial f_{\mu}}{\partial t}$$

$$\text{wegen } \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = \sum_{\mu=1}^M \sum_{i=1}^N \lambda_{\mu} \left(\frac{\partial f_{\mu}}{\partial \vec{r}_i} \right) \cdot \dot{\vec{r}}_i$$

$$\text{und } 0 = \frac{df_{\mu}}{dt} = \sum_{i=1}^N \vec{v}_i \cdot \frac{\partial f_{\mu}}{\partial \vec{r}_i} + \frac{\partial f_{\mu}}{\partial t}$$

$\Rightarrow$ Energie ist erhalten, wenn Zwangsbedingungen skleronom sind. $\left(\frac{\partial f_{\mu}}{\partial t} = 0 \right)$

Lagrange II & Newton: (Konservative Kräfte) (1 Punktformel)

$$T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m \sum_{j=1}^3 \dot{x}_j^2, \quad q_1 = x_1, \quad q_2 = x_2, \quad q_3 = x_3$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left[\left(\frac{1}{2} m \sum_{k=1}^3 \dot{q}_k^2 \right) - V(q_1, q_2, q_3) \right] - \frac{\partial}{\partial q_j} [T(q_1, q_2, q_3) - V(q_1, q_2, q_3)] = \frac{d}{dt} m \dot{q}_j + \frac{\partial V}{\partial q_j} = 0$$

$$\Leftrightarrow m \ddot{q}_j = - \frac{\partial V}{\partial q_j} \quad j=1, 2, 3$$

2.5 Lagrangian equation of 2nd kind

idea: express $\underline{r}_i, \dot{\underline{r}}_i, \underline{F}_i$ by generalized coordinates $q_j, \dot{q}_j, \delta q_j, j=1, \dots, f$
 $i=1, \dots, N$

$$\text{e.g.: } \underline{F}_i = \sum_{j=1}^f \left(\frac{\partial \underline{F}_i}{\partial q_j} \right) \delta q_j, \quad \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j} = \frac{\partial \underline{r}_i}{\partial q_j}, \quad \frac{d}{dt} \left(\frac{\partial \underline{r}_i}{\partial \dot{q}_j} \right) = \frac{\partial \underline{r}_i}{\partial q_j}$$

Defining the Lagrangian / Lagrange function: $L = T - V$

with $T = \sum_{i=1}^N \frac{1}{2} m_i \dot{\underline{r}}_i^2$ and $Q_j = - \frac{\partial V}{\partial q_j}$ (generalized forces):

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad j=1, \dots, f$$

$$\Leftrightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$(i) \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = \sum_{i=1}^N m \ddot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \frac{d}{dt} \left(\sum_{i=1}^N \frac{m_i}{2} \dot{\vec{x}}_i^2 \right) = \frac{d}{dt} T$$

$$(ii) \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i = - \sum_{i=1}^N \left(\vec{\nabla}_{\vec{r}_i} \sum_{j=1}^N \left(V_{ij}(|\vec{r}_{ij}|) + \sum_{k=1}^N V_i(|\vec{r}_i|) \right) \right) \cdot \dot{\vec{r}}_i$$

Konservative Kräfte

$$= - \frac{d}{dt} \left(\frac{1}{2} \sum_{i,j=1}^N V_{ij}(|\vec{r}_{ij}|) + \sum_{i=1}^N V_i(|\vec{r}_i|) \right)$$

$$= - \frac{d}{dt} V \quad \left(\frac{d}{dt} V_{ij}(|\vec{r}_{ij}|) = \left(\vec{\nabla}_{\vec{r}_i} V_{ij}(|\vec{r}_{ij}|) \right) \cdot \dot{\vec{r}}_i + \left(\vec{\nabla}_{\vec{r}_j} V_{ij}(|\vec{r}_{ij}|) \right) \cdot \dot{\vec{r}}_j \right)$$

$$i+ii \Rightarrow \frac{dT}{dt} + \frac{dV}{dt} = 0$$

(evtl. können dissipative Prozesse, z.B. Reibung, die Energieerhaltung stören/verfehlen)

$$\frac{d}{dt}(T+V) = \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i$$

Wenn Berücksichtigung von Zwangsbedingungen: $\sum_{i=1}^N \vec{z}_i \cdot \dot{\vec{r}}_i$ auf rechte Seite

$$\frac{d}{dt}(T+V) = - \sum_{\mu=1}^M \lambda_{\mu} \frac{\partial f_{\mu}}{\partial t}$$

$$\text{wegen } \sum_{i=1}^N \vec{z}_i \cdot \dot{\vec{r}}_i = \sum_{\mu=1}^M \sum_{i=1}^N \lambda_{\mu} \left(\vec{\nabla}_{\vec{r}_i} f_{\mu} \right) \cdot \dot{\vec{r}}_i$$

$$\text{und } 0 = \frac{df_{\mu}}{dt} = \sum_{i=1}^N \vec{\nabla}_{\vec{r}_i} f_{\mu} \cdot \dot{\vec{r}}_i + \frac{\partial f_{\mu}}{\partial t} \Rightarrow \frac{\partial f_{\mu}}{\partial t} = - \sum_{i=1}^N \left(\vec{\nabla}_{\vec{r}_i} f_{\mu} \right) \cdot \dot{\vec{r}}_i$$

$\Rightarrow$ Energie ist erhalten, wenn Zwangsbedingungen skleronom sind.
($\frac{\partial f_{\mu}}{\partial t} = 0$)

Lagrange II & Newton: (Konservative Kräfte) (1 Punktformale)

$$T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m \sum_{j=1}^3 \dot{x}_j^2, \quad q_1 = x_1, \quad q_2 = x_2, \quad q_3 = x_3$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left[\left(\frac{1}{2} m \sum_{k=1}^3 \dot{q}_k^2 \right) - V(q_1, q_2, q_3) \right] - \frac{\partial}{\partial q_j} [T(q_1, q_2, q_3) - V(q_1, q_2, q_3)]$$

$$= \frac{d}{dt} m \dot{q}_j + \frac{\partial V}{\partial q_j} = 0$$

$$\Leftrightarrow m \ddot{q}_j = - \frac{\partial V}{\partial q_j} \quad j=1, 2, 3$$

$$T(q_k, \dot{q}_k, t) = \sum_{i=1}^N \frac{1}{2} m_i \dot{r}_i(q_k, \dot{q}_k, t)^2$$

$$= \sum_{i=1}^N \frac{1}{2} m_i \left[\sum_{j=1}^f \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right]^2$$

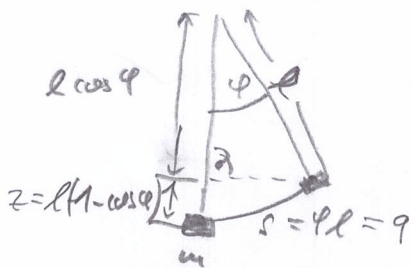
$$\stackrel{\uparrow}{=} a + \sum_{j=1}^f b_j \dot{q}_j + \sum_{j,m=1}^f c_{jm} \dot{q}_j \dot{q}_m$$

Sortieren nach
Ordnung in $\dot{q}_j$

Nach für skalaroname Zwangsbedingungen ist T eine konsequente
Bildungsaufbau ($a = b_j = 0$) in $\dot{q}_j$ ($j=1, \dots, f$)

2.6 Normalisierungsregeln

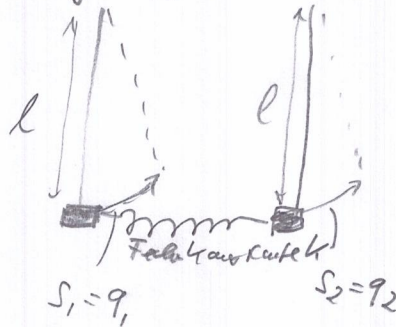
a) 1 Pendel



Nach Sichtweise:

- holonom, skalaroname Zwangsbedingungen

b) 2 gekoppelte Pendel



Annahme: Potenzial V mit stabiler Ruhelage

- Wahl der generalisierten Koordinaten $q_1, \dots, q_f$ mit Ruhelage $q_1 = \dots = q_f = 0$
- Entwicklung des Potentials um die Ruhelage:

$$V(q_1, \dots, q_f) = V(0, \dots, 0) + \sum_{j=1}^f \underbrace{\left(\frac{\partial V}{\partial q_j} \right)}_{\substack{\text{d. Balk} = 0 \\ \text{(Offset der Energie)}}} q_j + \frac{1}{2} \sum_{j,k=1}^f \underbrace{\left(\frac{\partial^2 V}{\partial q_j \partial q_k} \right)}_{\substack{= -Q_{jk} = 0 \\ \uparrow \\ \text{Ruhelage}}} q_j q_k + \dots$$

$$\Rightarrow V(q_1, \dots, q_f) \approx \frac{1}{2} \sum_{j,k=1}^f V_{jk} q_j q_k \geq 0 \quad \text{mit } V_{jk} = V_{kj}$$

pos. definite quadratische Form, da Ruhelage stabil & $T \geq 0$

• kinetische Energie: $T = \frac{1}{2} \sum_{j,k=1}^f T_{jk} \dot{q}_j \dot{q}_k \geq 0$ mit $T_{jk} = T_{kj} \approx \sum_{i=1}^N m_i \left(\frac{\partial r_i}{\partial q_j} \right) \left(\frac{\partial r_i}{\partial q_k} \right)$

Fall a) $T = \frac{1}{2} m \dot{q}^2$

$$V(q) = mg z(q) = mg l (1 - \cos \varphi(q)) = mg l (1 - \cos \frac{q}{l})$$

$$\Rightarrow \left. \begin{aligned} \left(\frac{\partial V}{\partial q} \right)_0 &= mg \frac{l}{l} \sin \frac{q}{l} \Big|_0 = 0 \\ \left(\frac{\partial^2 V}{\partial q^2} \right)_0 &= m \frac{g}{l} \cos \frac{q}{l} \Big|_0 = m \frac{g}{l} \end{aligned} \right\} \Rightarrow V(q) \approx \frac{1}{2} \left(\frac{\partial^2 V}{\partial q^2} \right)_0 q^2 = \frac{1}{2} m \frac{g}{l} q^2$$

$$L = T - V$$

$$\Rightarrow 0 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \stackrel{!}{=} \frac{d}{dt} (m \dot{q}) + m \frac{g}{l} q$$

$$= m \ddot{q} + m \frac{g}{l} q$$

$$\Leftrightarrow 0 = \ddot{q} + \frac{g}{l} q \quad \text{oder} \quad 0 = \ddot{q} + \omega^2 q \quad \text{mit} \quad \omega^2 = \frac{g}{l}$$

Fall b): $T = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2)$

$$V(q_1, q_2) = mg l (1 - \cos \frac{q_1}{l}) + mg l (1 - \cos \frac{q_2}{l}) + \frac{k}{2} (q_1 - q_2)^2$$

Hookesches Gesetz

$$\left(\frac{\partial^2 V}{\partial q_1^2} \right)_0 = \left(\frac{\partial^2 V}{\partial q_2^2} \right)_0 = m \frac{g}{l} + k$$

$$\left(\frac{\partial^2 V}{\partial q_1 \partial q_2} \right)_0 = mg \underbrace{\frac{\partial}{\partial q_1} \left(\sin \frac{q_2}{l} \right)}_{=0} - k^2 \frac{\partial}{\partial q_1} (q_1 - q_2) = -k = \left(\frac{\partial^2 V}{\partial q_2 \partial q_1} \right)_0$$

$$\Rightarrow V(q_1, q_2) \approx \frac{1}{2} m \frac{g}{l} (q_1^2 + q_2^2) + \frac{1}{2} k (q_1 - q_2)^2$$

in Matrixnotation:

$$\underline{\underline{I}} \equiv \left\{ \begin{matrix} T \\ \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 \end{matrix} \right\} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \left\{ \begin{matrix} V \\ \frac{1}{2} m \frac{g}{l} q_1^2 + \frac{1}{2} m \frac{g}{l} q_2^2 + \frac{k}{2} (q_1 - q_2)^2 \end{matrix} \right\} = \begin{pmatrix} m \frac{g}{l} + k & -k \\ -k & m \frac{g}{l} + k \end{pmatrix} \equiv \underline{\underline{V}}$$

Bewegungsgleichungen: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad j=1, 2$

$$\left. \begin{aligned} m \ddot{q}_1 + m \frac{g}{l} q_1 + k (q_1 - q_2) &= 0 \\ m \ddot{q}_2 + m \frac{g}{l} q_2 - k (q_1 - q_2) &= 0 \end{aligned} \right\} \text{Lösen gekoppelte DGLs}$$

$$\frac{d}{dt}(T+V) = - \underbrace{\sum_{\mu=1}^N \lambda_{\mu} \frac{\partial f_{\mu}}{\partial t}}$$

= 0 for scleronomic constraints

↳ $f_{\mu}(r_1, \dots, r_N, t)$ not explicitly time-dependent

2.6 Normal modes

a) 1 pendulum

b) 2 coupled pendula

idea: expand potential V at points of rest (set to $q_1 = q_2 = \dots = q_f = 0$)

↳ scaling of energy: $V(0, \dots, 0) = 0$

$$\hookrightarrow V(q_1, \dots, q_f) \approx \frac{1}{2} \sum_{j,k=1}^f \underbrace{V_{jk}}_{\left. \frac{\partial^2 V}{\partial q_j \partial q_k} \right|_0} q_j q_k \geq 0$$

• Kinetic energy:

$$\begin{aligned} T(q_k, \dot{q}_k, t) &= \sum_{i=1}^N \frac{1}{2} m_i \dot{r}_i(q_k, \dot{q}_k, t)^2 \\ &= \sum_{i=1}^N \frac{1}{2} m_i \left[\sum_{j=1}^f \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right]^2 \\ &= a + \sum_{j=1}^f b_j \dot{q}_j + \sum_{j,k=1}^f c_{jk} \dot{q}_j \dot{q}_k \end{aligned}$$

for scleronomic constraints $a = b_j = 0$

$$\begin{aligned} \Rightarrow T(\dot{q}_1, \dots, \dot{q}_f) &= \frac{1}{2} \sum_{j,k=1}^f T_{jk} \dot{q}_j \dot{q}_k \geq 0 \\ &\stackrel{\uparrow N}{\approx} \sum_{i=1}^N m_i \left(\frac{\partial r_i}{\partial q_j} \right)_0 \left(\frac{\partial r_i}{\partial q_k} \right)_0 \end{aligned}$$

a) $T = \frac{1}{2} m \dot{q}^2, V = \frac{1}{2} m \frac{g}{l} q^2$

$$\Rightarrow 0 = \ddot{q} + \frac{g}{l} q$$

$$b) T = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2)$$

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$$V = \frac{1}{2} m \frac{g}{\ell} (q_1^2 + q_2^2) + \frac{1}{2} k (q_1 - q_2)^2$$

matrix form: $T = \left\{ T_{jk} \right\}_{j,k=1,2} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

$$V = \left\{ V_{jk} \right\}_{j,k=1,2} = \begin{pmatrix} m \frac{g}{\ell} + k & -k \\ -k & m \frac{g}{\ell} + k \end{pmatrix}$$

equations of motion: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, j=1,2$

$$m \ddot{q}_1 + m \frac{g}{\ell} q_1 + k (q_1 - q_2) = 0$$

$$m \ddot{q}_2 + m \frac{g}{\ell} q_2 - k (q_1 - q_2) = 0$$

using the ansatz: $q_j = A_j e^{i\omega t}$:

eigenvalue equation:
$$\underbrace{\begin{pmatrix} m \frac{g}{\ell} + k & -k \\ -k & m \frac{g}{\ell} + k \end{pmatrix}}_{\underline{V}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \underbrace{m\omega^2}_{\omega^2 \underline{T}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Lösungssatz:

$$q_j = A_j e^{i\omega t} \Rightarrow \ddot{q}_j = -\omega^2 A_j e^{i\omega t}$$

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$$\Rightarrow \text{Eigenwertgleichung: } \begin{pmatrix} m \frac{g}{l} + k - m\omega^2 & -k \\ -k & m \frac{g}{l} + k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$\Rightarrow$ charakteristische Gleichung:

$$0 = \det \begin{vmatrix} \frac{g}{l} + \frac{k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} - \omega^2 \end{vmatrix} = \omega^4 - 2\omega^2 \frac{g}{l} + \left(\frac{g}{l} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2$$

$$\Rightarrow \omega_{1,2}^2 = \left(\frac{g}{l} + \frac{k}{m}\right) \pm \frac{k}{m} = \begin{cases} \frac{g}{l} \\ \frac{g}{l} + 2\frac{k}{m} \end{cases}$$

Eigenfrequenzen: $\omega_1 = \sqrt{\frac{g}{l}}$: ungestörte Eigenfrequenz

$$\omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}} =: \sqrt{\omega_0^2 + 2\tilde{\omega}^2}$$

Eigenvektoren:

$$\text{zu } \omega_1: kA_1^{(1)} - kA_2^{(1)} = 0 \Rightarrow \begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \xrightarrow{\text{Normierung}} \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{Normierung: } m \sum_k |A_k|^2 = 1 \Rightarrow \sum_{j,k} A_j \cdot \frac{1}{\sqrt{m}} \frac{1}{\sqrt{m}} A_k$$

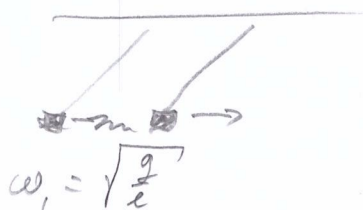
$$\text{zu } \omega_2: -kA_1^{(2)} - kA_2^{(2)} = 0 \Rightarrow \begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \xrightarrow{\text{Normierung}} \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{allgemeine Lsg: } \begin{pmatrix} A_1^{(1)} & A_2^{(1)} \\ A_1^{(2)} & A_2^{(2)} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} q_1 + q_2 \\ q_1 - q_2 \end{pmatrix} \leftarrow \begin{matrix} \text{Schwerpunkt-Koordinate} \\ \text{Relative Koordinate} \end{matrix}$$

Normalisierung:

$$= \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \Rightarrow \text{Umkehrung noch möglich}$$

$$q_j(t) \sim A_j^{(1)} Q_1 + A_j^{(2)} Q_2$$



$$\omega_1 = \sqrt{\frac{g}{l}}$$

gleichphasig

$\Rightarrow$ Überlagern: Schwebung



$$\omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}}$$

gegenphasig

Normalisierung ganz allgemein:

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$$L = T - V = \frac{1}{2} \sum_{j,k=1}^f (T_{jk} \dot{q}_j \dot{q}_k - V_{jk} q_j q_k)$$

$$\begin{aligned} \hookrightarrow \frac{\partial L}{\partial \dot{q}_l} &= \frac{1}{2} \sum_{j,k=1}^f T_{jk} \frac{\partial}{\partial \dot{q}_l} (\dot{q}_j \dot{q}_k) = \frac{1}{2} \sum_k T_{lk} \dot{q}_k + \frac{1}{2} \sum_j T_{jl} \dot{q}_j = \sum_k T_{lk} \dot{q}_k \\ &= \frac{\partial \dot{q}_j}{\partial \dot{q}_l} \dot{q}_k + \dot{q}_j \frac{\partial \dot{q}_k}{\partial \dot{q}_l} \\ &= \delta_{jl} \dot{q}_k + \dot{q}_j \delta_{kl} \\ &= \sum_{j,k} T_{jk} \dot{q}_k \end{aligned}$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_l} \right) = \sum_k T_{lk} \ddot{q}_k$$

$$\frac{\partial V}{\partial q_l} = \frac{1}{2} \sum_{j,k=1}^f T_{jk} \frac{\partial}{\partial q_l} (q_j q_k) = \dots = \sum_k V_{lk} q_k$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_l} \right) - \frac{\partial L}{\partial q_l} = 0 \Leftrightarrow \sum_k (T_{lk} \ddot{q}_k + V_{lk} q_k) = 0, \quad l=1, \dots, f$$

Lineare DGLs!

Ansatz: $q_k(t) = A_k e^{i\omega t}$

$$\Rightarrow \sum_{k=1}^f (V_{lk} - \omega^2 T_{lk}) A_k = 0$$

- Eigenwertgleichung für ω^2 ! $\omega_\alpha^2, \alpha=1, \dots, f$
- kann gelöst werden, liefert DGLs für $A_k \Rightarrow A_k^{(\alpha)}$

Nichttriviale Lsg genau dann, wenn $\det(V_{lk} - \omega^2 T_{lk}) = 0$

$$\text{allg. Lsg: } q_k(t) = \text{Re} \left\{ \sum_{\alpha=1}^f c_\alpha A_k^{(\alpha)} e^{i\omega_\alpha t} \right\}$$

Bestimmung durch Anfangsbed. $q_k(0), \dot{q}_k(0)$

V, T pos definit $\Rightarrow \omega^2 > 0$

$$\left[\sum_{k=1}^f (V_{lk} - \omega^2 T_{lk}) A_k = 0 \quad | \cdot \sum_{l=1}^f A_l^* \right]$$

$$\sum_{l,k=1}^f V_{lk} A_l^* A_k - \omega^2 \sum_{l,k=1}^f T_{lk} A_l^* A_k = 0 \Rightarrow \omega^2 = \frac{\sum_{l,k=1}^f V_{lk} A_l^* A_k}{\sum_{l,k=1}^f T_{lk} A_l^* A_k}$$

Idee: Transformieren auf neue Koordinaten Q_α , sodass die

Bewegungsgleichungen entkoppeln $\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0, \alpha = 1, \dots, f$

$\Rightarrow$ Hauptachsenstrafo der symmetrischen Matrizen $V_{\alpha\beta}, T_{\alpha\beta}$

$\Rightarrow$ Diagonalisierung durch Trafo mit (reell gewählten) Eigenvektoren $A_k^{(\alpha)}$:

$$q_k(t) = \sum_{\alpha=1}^f A_k^{(\alpha)} Q_\alpha(t) \Rightarrow \underline{q} = \underline{A} \underline{Q}$$

Check Diagonalform von $V_{\alpha\beta}, T_{\alpha\beta}$ (Gleichzeitdiagonalisierbarkeit)

$$I: \sum_{k=1}^f (V_{\alpha k} - \omega_\alpha^2 T_{\alpha k}) A_k^{(\alpha)} = 0 \quad | \cdot \sum_{l=1}^f A_l^{(\alpha)}$$

$$II: \sum_{l=1}^f (V_{\beta l} - \omega_\beta^2 T_{\beta l}) A_l^{(\beta)} = 0 \quad | \cdot \sum_{k=1}^f A_k^{(\alpha)}$$

$$\Rightarrow I-II: \sum_{k,l=1}^f \underbrace{(V_{\alpha k} - V_{\beta k}) A_k^{(\alpha)} A_l^{(\beta)}}_{=0} - (\omega_\alpha^2 T_{\alpha k} - \omega_\beta^2 T_{\beta k}) A_l^{(\beta)} A_k^{(\alpha)} = 0$$

$$\Rightarrow (\omega_\alpha^2 - \omega_\beta^2) \sum_{k=1}^f A_l^{(\beta)} T_{\alpha k} A_k^{(\alpha)} = 0$$

$\neq 0$ für $\alpha \neq \beta$ $\quad \tilde{T}^{(\alpha)\beta} \stackrel{!}{=} 0$ für $\alpha \neq \beta$ (trans Sonante kinetische Energie)

$$\Rightarrow \sum_{k=1}^f A_l^{(\beta)} T_{\alpha k} A_k^{(\alpha)} = \delta_{\beta\alpha} \quad (\text{geeignete Normierung der Eigenvektoren } A_k \text{ für } \alpha=\beta)$$

$$\Rightarrow \sum_{k=1}^f A_l^{(\beta)} V_{\alpha k} A_k^{(\alpha)} = \omega_\alpha^2 \sum_{k=1}^f A_l^{(\beta)} T_{\alpha k} A_k^{(\alpha)} = \omega_\alpha^2 \delta_{\beta\alpha}$$

$\Rightarrow T_{\alpha\beta}$ & $V_{\alpha\beta}$ gleichzeitig diagonalisierbar

$$\Rightarrow L = \frac{1}{2} \sum_{j,k=1}^f (\tilde{T}_{jk} \dot{q}_j \dot{q}_k - V_{jk} q_j q_k)$$

$$= \frac{1}{2} \sum_{\alpha,p=1}^f \underbrace{\left(\sum_{j,k=1}^f A_j^{(\beta)} \tilde{T}_{jk} A_k^{(\alpha)} \right)}_{\delta_{\beta\alpha}} \dot{Q}_\alpha \dot{Q}_\beta - \sum_{j,k=1}^f A_j^{(\beta)} V_{jk} A_k^{(\alpha)} Q_\alpha Q_\beta$$

$$= \frac{1}{2} \sum_{\alpha=1}^f (\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2) \Rightarrow \text{Entkopplung: } \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Q}_\alpha} - \frac{\partial \mathcal{L}}{\partial Q_\alpha} = 0 = \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$$

2.6 Normal modes (continued)

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case: 2 coupled pendula

$$\left. \begin{aligned} m \ddot{q}_1 + m \frac{g}{l} q_1 + k(q_1 - q_2) &= 0 \\ m \ddot{q}_2 + m \frac{g}{l} q_2 - k(q_1 - q_2) &= 0 \end{aligned} \right\} \text{derived via Lagrange II}$$

transform into eigenvalue / vector problem:

Eigenvalues: $\omega_1^2 = \frac{g}{l}$

Eigenvectors: $\begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

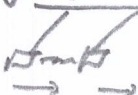
$\omega_2^2 = \frac{g}{l} + 2 \frac{k}{m}$

$\begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ Solutions: $Q_1 = \frac{1}{\sqrt{2m}} (q_1 + q_2)$: motion of center of mass

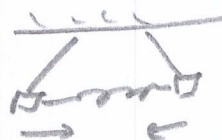
normal modes

in-phase



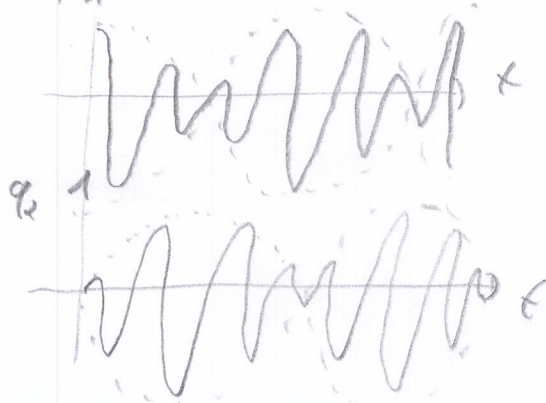
$Q_2 = \frac{1}{\sqrt{2m}} (q_1 - q_2)$: relative motion

anti-phase



$$(q_j(t)) = \sum_{\alpha} A_j^{(\alpha)} Q_{\alpha}(t)$$

General solution: mode beating



general approach/case: $\mathcal{L} = T - V = \frac{1}{2} \sum_{j,k=1}^f \left(T_{jk} \dot{q}_j \dot{q}_k - V_{jk} q_j q_k \right)$

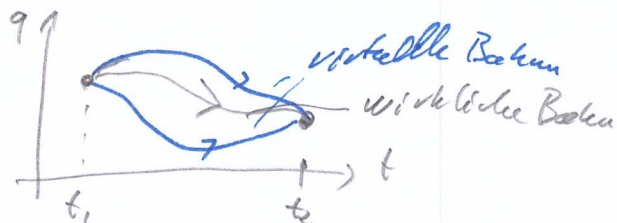
$$\sum_{k=1}^f (V_{jk} - \omega^2 T_{jk}) A_k = 0 \Rightarrow \det(V_{jk} - \omega^2 T_{jk}) = 0 \Rightarrow \omega_1^2, \dots, \omega_f^2$$

General solution: $q_k(t) = \text{Re} \left\{ \sum_{\alpha=1}^f \underset{\substack{\uparrow \\ \text{arbitrary boundary condition}}}{C_{\alpha}} A_k^{(\alpha)} e^{i\omega_{\alpha} t} \right\}, \ddot{Q}_{\alpha} + \omega_{\alpha}^2 Q_{\alpha} = 0$

3.1 Variationsprinzip

Bisher: differenzierbare Variation $\delta r_i \rightarrow$ D'Alemberts Prinzip

Nun: Variation der ganzen Bahn $\rightarrow$ ~~Hamiltons~~ **Hamiltonsches Prinzip**



Idee: Die wirklich ausgeführte Bahn zeichnet sich aus dadurch aus, dass sie eine konstante Größe ("Wirkung") extremal macht.

Allgemein: Sei $I: C^2 \rightarrow \mathbb{R}$ eine Funktional

$$q(t) \mapsto I[q] := \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt$$

Suche $q(t)$, so dass $\delta I[q] = 0$ (extremal!)

Annahme für varierte Bahnen:

Zu jedem t aus $t_1 \leq t \leq t_2$ wird dem Punkt $q(t)$ auf der realen Bahn ein variierter Bahnpunkt $q'(t)$ zugeordnet:

(i) $q'(t) \in C^2$ (2mal differenzierbar)

(ii) $\delta q(t) = q'(t) - q(t)$ differenzierbare Variation

(iii) $\delta t = 0$ Zeit nicht variert

(iv) $q'(t_1) = q(t_1)$
 $q'(t_2) = q(t_2)$

Aufgabe: Punkte fest $\Rightarrow \delta q(t_1) = \delta q(t_2) = 0$

a) Extremum einer Funktion $f(x)$ einer Variable:

$$\delta f(x) = f(x+\delta x) - f(x) = \frac{df}{dx} \delta x = 0 \quad \text{für beliebige } \delta x \neq 0$$

$$\Rightarrow \frac{df}{dx} = 0 \quad \text{bei } x = x^* (\text{Extremum})$$

b) Extremum einer Funktion $f(x_1, \dots, x_N)$ mehrerer Variablen:

$$\delta f = f(x_1 + \delta x_1, \dots, x_N + \delta x_N) - f(x_1, \dots, x_N) = \sum_{i=1}^N \frac{\partial f}{\partial x_i} \delta x_i = 0 \quad \text{für bel. } \delta x_i$$

$$\Rightarrow \frac{\partial f}{\partial x_i} = 0 \quad i=1, \dots, N \quad \text{bei } x_i = x_i^* (\text{Extremum})$$

Funktion

c) Extremum eines **Funktional** $f[x] \equiv f[x(t)] \quad (= f(x(t)))$

$$x_1, \dots, x_N \rightarrow x(t)$$

$$x_i^* \rightarrow x^*(t)$$

$$\delta x_1, \dots, \delta x_N \rightarrow \delta x(t)$$

$$\delta f = \sum_{i=1}^N \frac{\partial f}{\partial x_i} \delta x_i \rightarrow \delta f = \int_{t_1}^{t_2} \frac{\delta f}{\delta x(t)} \delta x(t)$$

Funktionalableitung

$$\hookrightarrow f[x] = \int_{t_1}^{t_2} dt \, F(x(t))$$

$$\Rightarrow \delta f = f[x(t) + \delta x(t)] - f[x(t)] = \int_{t_1}^{t_2} dt \{ F(x(t) + \delta x(t)) - F(x(t)) \}$$

$$= \int_{t_1}^{t_2} dt \underbrace{\frac{dF}{dx}}_{\frac{\delta f}{\delta x(t)}} \delta x(t) = 0 \quad \text{für beliebige } \delta x(t)$$

$$\Rightarrow \frac{\delta f}{\delta x(t)} = \frac{dF}{dx} = 0 \quad \text{bei } x(t) = x^*(t) (\text{Extremum})$$

partielle Integration

$$\text{Bsp: } f[x, \dot{x}] = \int_{t_1}^{t_2} dt \, F(x(t), \dot{x}(t))$$

$$\Rightarrow \delta f = \int_{t_1}^{t_2} dt \, \delta F(x(t), \dot{x}(t)) = \int_{t_1}^{t_2} dt \left\{ \frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial \dot{x}} \delta \dot{x} \right\} = \int_{t_1}^{t_2} dt \left\{ \frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} \right\} \delta x(t)$$

$$\Rightarrow \text{Extremum: } \frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0 \quad (\text{Euler-Lagrange-Gleichung für } x(t))$$

Anwenden auf $q(t)$, $\bar{T}(q, \dot{q}, t)$:

$$0 = \delta I[q] = \int_{t_1}^{t_2} dt \bar{T}(q, \dot{q}, t) \stackrel{(iv)}{=} \int_{t_1}^{t_2} dt \delta \bar{T}(q, \dot{q}, t) \stackrel{(iii)}{=} \int_{t_1}^{t_2} dt \left[\frac{\partial \bar{T}}{\partial q} \delta q + \frac{\partial \bar{T}}{\partial \dot{q}} \delta \dot{q} \right]$$

$$= \int_{t_1}^{t_2} dt \frac{\partial \bar{T}}{\partial q} \delta q + \int_{t_1}^{t_2} dt \frac{\partial \bar{T}}{\partial \dot{q}} \delta \dot{q}$$

$$\stackrel{\text{partielle Int.}}{=} \left. \frac{\partial \bar{T}}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \left(\frac{d}{dt} \frac{\partial \bar{T}}{\partial \dot{q}} \right) \delta q$$

partielle Int. $\int \frac{d}{dt} \frac{\partial \bar{T}}{\partial \dot{q}} \delta q = 0$

$$= \int_{t_1}^{t_2} dt \left\{ \frac{\partial \bar{T}}{\partial q} - \frac{d}{dt} \frac{\partial \bar{T}}{\partial \dot{q}} \right\} \delta q$$

frei wählbar

$$\Rightarrow \frac{\partial \bar{T}}{\partial q} - \frac{d}{dt} \frac{\partial \bar{T}}{\partial \dot{q}} = 0$$

Fazit: Differentialgleichung ist äquivalent zum Integralprinzip $\delta I[q] = 0$

3.2 Hamiltonsches Wirksprinzip

Annahmen:

(i) Holonome Zwangsbedingungen $\rightarrow$ generalisierte Koord. $q_1, \dots, q_f$

(ii) Konserватive Kräfte $\rightarrow$ Lagrange-Funktion $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t) = T - V$

Idee: (a) $T = L$ (b) Verallgemeinerung auf vektorielle Variablen $x \rightarrow q_1, \dots, q_f$

$\Rightarrow$ Euler-Lagrange - DGL $\equiv$ Lagrange-Gleichungen 2. Art $\dot{x} \rightarrow \dot{q}_1, \dots, \dot{q}_f$

Integralprinzip $\delta I = 0 \equiv$ Hamiltonsches Wirksprinzip $\delta W = 0$

Wirkungsfunktional $W = \int_{t_1}^{t_2} dt L(q_1(t), \dots, q_f(t), \dot{q}_1(t), \dots, \dot{q}_f(t), t)$

Berechne:

$$\begin{aligned} \mathcal{O} = \int W &= \int_{t_1}^{t_2} dt \sum_{k=1}^f \left\{ \frac{\partial \mathcal{L}}{\partial q_k} \dot{q}_k(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \frac{d}{dt} \dot{q}_k(t) \right\} \\ &= \int_{t_1}^{t_2} dt \sum_{k=1}^f \left\{ \frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right\} \dot{q}_k(t) \end{aligned}$$

Für beliebige Variationen $\delta q_k(t)$, $k=1, \dots, f$ folgt:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

Bsp: 1-dimensionaler harmonischer Oszillator

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \\ \Rightarrow \mathcal{O} = \int W &= \int_{t_1}^{t_2} dt \int \left\{ \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right\} \\ &= \int_{t_1}^{t_2} dt \left\{ m \dot{q} \delta \dot{q} - m \omega^2 q \delta q \right\} \\ &= \int_{t_1}^{t_2} dt \left\{ -m \ddot{q} - m \omega^2 q \right\} \delta q \\ &\stackrel{\text{partielle Int.}}{=} \int_{t_1}^{t_2} dt \left\{ -m \ddot{q} - m \omega^2 q \right\} \delta q \\ \Rightarrow \ddot{q} + \omega^2 q &= 0 \end{aligned}$$

Bem.: • Integralprinzip (betrifft ganze Bahn)

- **teleologisch**: auf ein Ziel gerichtet
- unabhängig von Koordinatenwahl

$$[Wirkung] = \underbrace{[Energie]}_{\frac{kg}{s^2}} \times \underbrace{[Zeit]}_s = \underbrace{[Impuls]}_{\frac{kg}{s}} \times \underbrace{[L.]}_m$$

Vgl. Plancks Wirkungsquantum $[\hbar] = J \cdot s$ ($\hbar = 6,626 \cdot 10^{-34} J \cdot s$)
 $(\hbar = \frac{h}{2\pi} = 1,055 \cdot 10^{-34} J \cdot s)$

3.3 Eichtransformationen

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Lagrange-Funktion nicht eindeutig durch Lagrange-Gleichung festgelegt.

Satz: Die Eichtransformationen $L \mapsto L' = L + \frac{d}{dt} M(q, t)$ mit beliebiger
Eichfunktion M löst die Lagrange-Gleichung *invariant*.

Folgerung: $\oint L' dt = 0 \Leftrightarrow \oint L dt = 0$

Beweis: $\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}_k} \right) - \frac{\partial L'}{\partial q_k} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} + \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_k} \frac{d}{dt} M(q, t) \right) - \frac{\partial}{\partial q_k} \left(\frac{d}{dt} M(q, t) \right)$

mit $\frac{d}{dt} M(q, t) = \sum_{l=1}^f \frac{\partial M}{\partial q_l} \dot{q}_l + \frac{\partial M}{\partial t}$ folgt:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}_k} \right) - \frac{\partial L'}{\partial q_k} &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} + \frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} \left(\sum_{l=1}^f \frac{\partial M}{\partial q_l} \dot{q}_l + \frac{\partial M}{\partial t} \right) - \frac{\partial}{\partial q_k} \left(\frac{d}{dt} M(q, t) \right) \\ &= \frac{\partial M}{\partial q_k} \quad (\text{M darf nicht von } \dot{q}_k \text{ abhängen}) \\ &= 0 \end{aligned}$$

Bsp.: 1D harmonischer Osz.

$$L = \frac{m}{2} \dot{q}^2 - \frac{m}{2} \omega^2 q^2 \quad \text{und} \quad M(q) = \frac{m}{2} \omega^2 q^2 \Rightarrow \frac{d}{dt} M = m \omega^2 q \dot{q}$$

$$\Rightarrow L = \frac{m}{2} \dot{q}^2 - \frac{m}{2} \omega^2 (q^2 - 2q\dot{q})$$

$$\Rightarrow 0 = \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}} \right) - \frac{\partial L'}{\partial q} = m\ddot{q} + m\omega^2 \dot{q} - (-m\omega^2 q + m\omega^2 \dot{q}) = \ddot{q} + \omega^2 q = 0$$

3 Hamiltonian principle

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3.1 variational problem

D'Alembert principle $\leftrightarrow$ differential variation (virtual displacements δq_i)

Hamiltonian principle $\leftrightarrow$ variation of entire trajectory

- Define an action functional $W: C^2 \rightarrow \mathbb{R}$
 $q(t) \mapsto W[q] := \int_{t_1}^{t_2} dt L(q, \dot{q}, t)$

such that $\delta W[q] = 0$ for the real $q(t)$.

- Variations $q'(t)$ of $q(t)$: $t_1 \leq t \leq t_2$

- (i) $q'(t) \in C^2$ (ii) $\delta q(t) = q'(t) - q(t)$ (iii) $\delta t = 0$ (iv) $q'(t_1) = q(t_1)$
 $q'(t_2) = q(t_2)$

3.2 Hamiltonian principle

- $0 = \delta W[q] = \int_{t_1}^{t_2} dt \sum_{k=1}^f \left\{ \frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \delta q_k \right\}$

$$= \int_{t_1}^{t_2} dt \sum_{k=1}^f \left\{ \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right\} \delta q_k$$

$$\Rightarrow \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0 \quad k=1, \dots, f \quad (\text{Euler-Lagrange equations})$$

3.3 Gauge transformation

Theorem: The gauge transformation $L \rightarrow L' = L + \frac{d}{dt} M(q, t)$ leaves the Euler-Lagrange equations invariant.

4 Symmetrie & Erhaltungssätze

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4.1 Noether - Theorem

4.2 Räumliche Translationsinvarianz

4.3 Rotationsinvarianz (Isotropie)

4.4 Zeitliche Translationsinvarianz

4.5 Das Zweikörperproblem

4.1 Noether - Theorem

Idea: Kontinuierliche Transformation, unter denen das physikalische System ^{invariant} invariant ist, entsprechen **Erhaltungsgrößen**

(**Integrale der Bewegung, Konstanten der Bewegung**)

- Voraussetzung: **autonom** (nicht explizit zeitabhängig) System mit f Freiheitsgraden und Lagrange-Funktion $L(q_i, \dot{q}_i, t)$

Theorem von Emmy Noether:

Die Lagrange-Funktion $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$ eines autonomen Systems sei unter der Transformation $q \rightarrow \underline{h}^S(q)$ invariant (S : kontinuierlicher Parameter und $\underline{h}^{S=0}(q) = q$). Dann gibt es ein Integral der Bewegung

$$I(q, \dot{q}) = \sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \left(\frac{d}{ds} \underline{h}_j^S(q_j) \right) \Big|_{S=0} \quad \text{*) } \underline{h}^S(q) = q + S \frac{d\underline{h}^S(q)}{ds} \Big|_{S=0} + \mathcal{O}(S^2)$$

Beweis: $q = q(t)$ Lösung der Lagrange-Gleichungen

$$\Rightarrow q(s, t) = \underline{h}^S(q, t) \quad \text{,,} \quad \text{weil invariant.}$$

$$(i) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j}, \quad (ii) \frac{d}{ds} L(q(s, t), \dot{q}(s, t)) = \sum_{j=1}^f \left(\frac{\partial L}{\partial q_j} \frac{dq_j}{ds} + \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{ds} \right) = 0$$

$$\Rightarrow \frac{d}{dt} I(q, \dot{q}) = \sum_{j=1}^f \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_j} \left(\frac{d}{ds} \underline{h}_j^S(q_j) \right) \Big|_{S=0} \right] = \sum_{j=1}^f \left[\underbrace{\left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \right)}_{= \frac{\partial L}{\partial q_j}} \frac{dq_j}{ds} + \frac{\partial L}{\partial \dot{q}_j} \underbrace{\frac{d}{dt} \left(\frac{d\dot{q}_j}{ds} \right)}_{= \frac{d\ddot{q}_j}{ds}} \right]$$

$$\stackrel{(ii)}{=} \frac{d}{ds} L \stackrel{\text{invariant}}{=} 0$$

$$\text{*) } \frac{d\underline{h}^S(q)}{ds} = \frac{dq}{ds} + \frac{d}{ds} \frac{d\underline{h}^S(q)}{ds} \Big|_{S=0} + \mathcal{O}(S^2)$$

• Konervative Kräfte, keine Zwangsbedingungen: $L = \frac{1}{2} \sum_{i=1}^N m_i \dot{\underline{r}}_i^2 - V(\underline{r}_1, \dots, \underline{r}_N)$

• Translations in $\underline{e}_x$ -Richtung: $\underline{h}^s: \underline{r}_i \rightarrow \underline{r}_i + s \underline{e}_x \quad i=1, \dots, N$

$\Rightarrow$ **Translationsinvarianz** (entlang x-Achse)

$$\Rightarrow L(\underbrace{\underline{h}^s(\underline{r}_i), \dot{\underline{r}}_i}_{(\underline{h}^s(\underline{r}_1), \dots, \underline{h}^s(\underline{r}_N), \dot{\underline{r}}_1, \dots, \dot{\underline{r}}_N)}) = \frac{1}{2} \sum_{i=1}^N m_i \underbrace{\left[\frac{d}{ds} (\underline{r}_i + s \underline{e}_x) \right]^2}_{\left[\frac{d}{ds} (\underline{r}_i) \right]^2 = \dot{\underline{r}}_i^2} - V(\underline{r}_i + s \underline{e}_x + \dots) = L(\underline{r}_i, \dot{\underline{r}}_i)$$

$$\Rightarrow \frac{dL}{ds} = - \sum_{i=1}^N (\underline{\nabla}_{\underline{r}_i} \cdot \underline{e}_x) V = - \sum_{i=1}^N \frac{\partial V}{\partial x_i} \stackrel{!}{=} 0 \quad (\text{Keine äußere Kraft in x-Richtung})$$

Integral der Bewegung:

$$I = \sum_{i=1}^N \underline{\nabla}_{\underline{r}_i} L \frac{d\underline{h}}{ds} \bigg|_{s=0} = \sum_{i=1}^N m_i \dot{\underline{r}}_i \cdot \underline{e}_x = \sum_{i=1}^N m_i \dot{x}_i = P_x \quad (\text{x-Komponente des Gesamtimpulses})$$

Translationsinvarianz in x-Richtung $\rightarrow$ Erhaltung des x-Komponente des Gesamtimpulses

• **Konjugierte verallgemeinl. Impulse** $p_j = \frac{\partial L}{\partial \dot{q}_j}$

Wähle $q_1 = s$ als verallgemeinl. Koordinate! $\underline{r}_i = \underline{r}_i(q_1, \dots, q_f, t) = q_1 \underline{e}_x + s \underline{r}_i(q_2, \dots, q_f, t)$

$$\Rightarrow \frac{\partial}{\partial q_1} \underline{r}_i = \underline{e}_x, \quad \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_1} = \frac{\partial}{\partial \dot{q}_1} \underline{r}_i = \underline{e}_x$$

$$\underline{\dot{r}}_i = \sum_{j=1}^f \frac{\partial \underline{r}_i}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial \underline{r}_i}{\partial t}$$

$$\Rightarrow \frac{\partial L}{\partial q_1} = 0 \quad (\Leftrightarrow) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} = 0 \quad (\Rightarrow) \quad \frac{\partial L}{\partial \dot{q}_1} = \text{const}$$

Invarianz

Erhaltungssatz

• $\frac{\partial L}{\partial q_1} = 0$ dann heißt q_1 **zyklische Koordinate** (und ihr konjugierter Impuls ist Erhaltungssatz)

Bsp: (i) Teilchen für Potenzial $V(\underline{r}) = V(x, z)$

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$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x} = P_x = \text{const}$$

$$\text{Integral der Bewegung: } \underbrace{\mathcal{L}}_{\substack{\nabla \cdot \mathcal{L} \\ = \underline{e}_x}} = \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{d\dot{x}}{ds} = \frac{\partial \mathcal{L}}{\partial \dot{x}} = P_x$$

(ii) 2 Teilchen mit Poisson Wechselwirkung: $V(\underline{r}_1, \underline{r}_2) = V(|\underline{r}_1 - \underline{r}_2|)$

Traslativ, invariant unter x, y, z -Richtung:

$$\underline{h}^S(\underline{r}_i) = \underline{r}_i + \sum_{j=x,y,z} s_j \underline{e}_j$$

$\Rightarrow$ Übungsblatt 5

4.1 Noether - Theorem

idea: continuous infinitesimal transformations that leave the physical system invariant ($\frac{dL}{ds} = 0$) $\Leftrightarrow$ conserved quantity

Theorem: Let the Lagrange function/Lagrangian $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$ of an autonomous system be invariant w.r.t. infinitesimal transformations $q \rightarrow h^s(q)$ (s : continuous parameter, $h^{s=0}(q) = q$)
Then there exists an integral of motion / conserved quantity:

$$I(q, \dot{q}) = \sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \left(\frac{d}{ds} h_j^s(q) \right) \Big|_{s=0}$$

$$\bullet h^s(q) = \underbrace{h^{s=0}(q)}_{=q} + s \left(\frac{d}{ds} h^s(q) \right) \Big|_{s=0} + O(s^2)$$

4.2 Translational invariance

translational invariance $\Leftrightarrow$ conservation of momentum

$$\left. \begin{array}{l} \text{e.g.: } h^s = \underline{r}_i \rightarrow \underline{r}_i + s \underline{e}_x \\ \text{and } \frac{dL}{ds} = 0 \end{array} \right\} \Rightarrow \sum_i m_i \dot{x}_i = P_x = \text{const.}$$

x -component of total momentum.

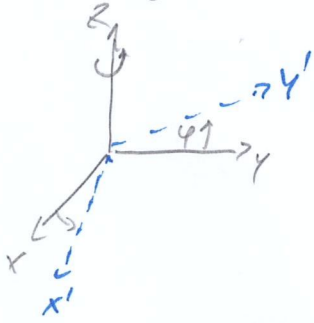
$$\bullet \text{ generalized momenta: } P_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$\bullet \text{ cyclic coordinates: } \frac{\partial L}{\partial \dot{q}_j} = 0 \Rightarrow P_j = \text{constant}$$

$\uparrow$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}_j}$$

Drehung um z-Achse (Winkel $\varphi = S$)

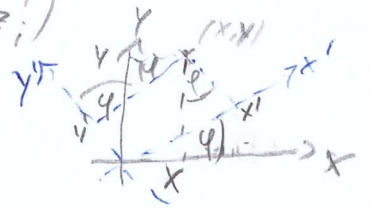


$$h^S: \underline{r}_i = (x_i, y_i, z_i) \rightarrow \underline{r}_i' = (x_i', y_i', z_i')$$

$$\text{mit } x_i' = x_i \cos S + y_i \sin S$$

$$y_i' = -x_i \sin S + y_i \cos S$$

$$z_i' = z_i$$



$$\Rightarrow \begin{pmatrix} x_i' \\ y_i' \\ z_i' \end{pmatrix} = \begin{pmatrix} \cos S & \sin S & 0 \\ -\sin S & \cos S & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \approx \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & S & 0 \\ -S & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

$$\textcircled{2} \begin{cases} x = r \cos \alpha \\ y = r \sin \alpha \end{cases} \Rightarrow \begin{cases} x' = r \cos(\alpha - S) = r \cos \alpha \cos S + r \sin \alpha \sin S \\ y' = r \sin(\alpha - S) = r \sin \alpha \cos S - r \cos \alpha \sin S \end{cases}$$

Erzeugende für infinitesimal

Drehung um z-Achse $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

xx
Koordinaten-
systeme

$$\Rightarrow \begin{cases} x' = x \cos S + y \sin S \\ y' = -x \sin S + y \cos S \end{cases}$$

$$= \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} + S \begin{pmatrix} y_i \\ -x_i \\ 0 \end{pmatrix} = \underline{r}_i + S(\underline{r}_i \times \underline{e}_z)$$

$$\underline{r}_i' = \underline{h}^S(\underline{r}_i) \Big|_{S=0} + S \left(\frac{d}{ds} \underline{h}^S(\underline{r}_i) \right) \Big|_{S=0} + O(S^2)$$

$$\text{mit } \frac{d}{ds} \left(\underline{h}^S(\underline{r}_i) \right) \Big|_{S=0} = \underline{r}_i \times \underline{e}_z$$

T ist rotationsinvariant

(hängt nur von $|\underline{r}_i|$ ab)

(Drehung: orthogonale Transformation)

$$\Rightarrow L(\underline{r}_i, \dot{\underline{r}}_i) = L(\underline{r}_i' = \underline{h}^S(\underline{r}_i), \dot{\underline{r}}_i)$$

$$\Rightarrow \left(\frac{dL}{ds} \right) \Big|_{S=0} = - \left(\frac{dV}{ds} \right) \Big|_{S=0} = - \sum_{i=1}^N \underbrace{(\underline{r}_i' \cdot \underline{V})}_{-\underline{T}_i} \underbrace{\left(\frac{d\underline{r}_i'}{ds} \right) \Big|_{S=0}}_{\left(\frac{d\underline{h}^S}{ds} \right) \Big|_{S=0}} = \sum_{i=1}^N \underline{T}_i \cdot (\underline{r}_i \times \underline{e}_z)$$

$$= \underline{e}_z \cdot \sum_{i=1}^N \underline{T}_i \times \underline{r}_i = - \underline{e}_z \cdot \sum_{i=1}^N (\underline{r}_i \times \underline{T}_i) \stackrel{!}{=} 0$$

2. Zeile: Prinzipal

$$\Rightarrow \text{Integral der Bewegung: } I = \sum_{i=1}^N \frac{\partial L}{\partial \dot{\underline{r}}_i} \left(\frac{d\underline{h}^S}{ds} \right) \Big|_{S=0} = \sum_{i=1}^N m_i \dot{\underline{r}}_i \cdot (\underline{r}_i \times \underline{e}_z)$$

$$= \underline{e}_z \cdot \sum_{i=1}^N (m_i \dot{\underline{r}}_i \times \underline{r}_i) = - \underline{e}_z \cdot \sum_{i=1}^N (\underline{r}_i \times m_i \dot{\underline{r}}_i) = - \underline{L}_z$$

Drehimpuls L

Rotationsinvarianz $\Rightarrow$ Drehimpuls erhalten

• Alternative Betrachtungsweise:

Wähle $q_1 = \varphi$ als generalisierte Koordinate: $\underline{r}_i = \underline{r}_i(\varphi, q_2, \dots, q_k, t)$

mit $\frac{\partial \underline{r}_i}{\partial \varphi} = \left(\frac{d}{dt} \underline{h}^i(\underline{r}_i) \right)_{\dot{\varphi}=0} = \underline{r}_i \times \underline{e}_z$ (infinitesimale Drehung um z-Achse)

Invarianz:

Erhaltungssatz

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad (\Leftrightarrow) \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 0 \quad (\Leftrightarrow) \quad P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \text{const}$$

$$\hookrightarrow P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial T}{\partial \dot{\varphi}} = \frac{1}{2} \sum_{i=1}^N m_i \frac{\partial (\dot{\underline{r}}_i)^2}{\partial \dot{\varphi}} = \sum_{i=1}^N m_i \dot{\underline{r}}_i \frac{\partial \dot{\underline{r}}_i}{\partial \dot{\varphi}} = \sum_{i=1}^N m_i \dot{\underline{r}}_i (\underline{r}_i \times \underline{e}_z) = \sum_{i=1}^N m_i \dot{\underline{r}}_i \underline{r}_i \times \underline{e}_z = \frac{\partial \underline{L}}{\partial \dot{\varphi}}$$

$$= -\underline{e}_z \sum_{i=1}^N (\underline{r}_i \times m_i \dot{\underline{r}}_i) = -\underline{L}_z \quad (\text{z-Komponente des Drehimpulses})$$

Bsp: N Teilchen mit ~~innerer~~ paarweiser Wechselwirkung, die nur vom

Abstand abhängt: $V(\underline{r}_1, \dots, \underline{r}_N) = V(\underline{r}_{12}, \dots, \underline{r}_{N-1,N})$ mit $\underline{r}_{ij} = |\underline{r}_i - \underline{r}_j|$

Rotationen nur ausserhalb gegen Drehung um alle Achsen (effektiv: Winkel φ)

$$\frac{\partial V}{\partial \varphi} = \sum_{i,j=1}^N \frac{\partial V}{\partial r_{ij}} \cdot \frac{\partial r_{ij}}{\partial \varphi} \stackrel{!}{=} 0$$

$$\hookrightarrow = \frac{\partial}{\partial \varphi} [(\underline{r}_i - \underline{r}_j)(\underline{r}_i - \underline{r}_j)]^{\frac{1}{2}} = \frac{1}{r_{ij}} (\underline{r}_i - \underline{r}_j) \frac{\partial (\underline{r}_i - \underline{r}_j)}{\partial \varphi} = \frac{\underline{r}_i - \underline{r}_j}{r_{ij}} \left(\frac{\partial \underline{r}_i}{\partial \varphi} - \frac{\partial \underline{r}_j}{\partial \varphi} \right)$$

$\underline{r}_i \times \underline{e}_z \quad \underline{r}_j \times \underline{e}_z$

$$= \frac{1}{r_{ij}} (\underline{r}_i - \underline{r}_j) [(\underline{r}_i - \underline{r}_j) \times \underline{e}_z] = \frac{1}{r_{ij}} \underline{e}_z [(\underline{r}_i - \underline{r}_j) \times (\underline{r}_i \times \underline{r}_j)] = 0$$

$= 0$

$\Rightarrow$ Drehimpuls $\underline{L}$ ist Erhaltungsgröße.

Zeitliche Translationsinvarianz:

(i) Zwangsbedingungen nicht explizit t -abhängig

$$\Rightarrow r_i = r_i(q_1, \dots, q_f), \quad \frac{\partial r_i}{\partial t} = 0 \Rightarrow \dot{r}_i = \sum_{j=1}^f \frac{\partial r_i}{\partial q_j} \dot{q}_j$$

$$(ii) \frac{\partial L}{\partial t} = 0$$

• Beobachtung: $\sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j \stackrel{!}{=} \sum_{j=1}^f \left(\frac{\partial T}{\partial \dot{q}_j} \right) \dot{q}_j = \left(\sum_{j=1}^f m_j \dot{q}_j \dot{q}_j \right) \stackrel{!}{=} 2T \quad (*)$

$L = T(\dot{q}) - V(q)$
 $\frac{1}{2} \sum_{j=1}^f m_j \dot{q}_j^2$
 Achtung! (Euler-Satz)
 genau! Siehe 3.2a
 gilt nicht immer!

$$\frac{d}{dt} L = \sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial q_j} \dot{q}_j \right) + \frac{\partial L}{\partial t}$$

$$= \sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j \right)$$

$$= \frac{d}{dt} \left(\sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) \stackrel{(*)}{=} 2 \frac{dT}{dt}$$

$$\Rightarrow 0 = \frac{d}{dt} (2T - L) \stackrel{L=T-V}{=} \frac{d}{dt} (T + V) \Rightarrow T + V = \text{const.}$$

Zeittranslationsinvarianz $\Rightarrow$ Energieerhaltung

Stetigkeitsbedingung $\frac{\partial L}{\partial t} = 0$
 Zwangsbedingung

$$\Rightarrow E = T + V = \text{const.}$$

Kinetische Energie: Nachtrag

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} \sum_{j,k} T_{jk} \dot{q}_j \dot{q}_k$$

mit $T_{jk} = \sum_{i=1}^N m_i \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} \right) \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_k} \right)$ abhängig von $q_1, \dots, q_k$!

T ist homogen quadratische Funktion in $\dot{q}_1, \dots, \dot{q}_f$:

$$T(\lambda \dot{q}_1, \dots, \lambda \dot{q}_f) = \lambda^2 T(\dot{q}_1, \dots, \dot{q}_f)$$

$$\frac{\partial}{\partial \lambda} \Big|_{\lambda=1} \Rightarrow \sum_k \frac{\partial T}{\partial (\lambda \dot{q}_k)} \frac{\partial (\lambda \dot{q}_k)}{\partial \lambda} \Big|_{\lambda=1} = 2 \lambda T(\dot{q}_1, \dots, \dot{q}_f) \Big|_{\lambda=1}$$

$\underbrace{\qquad\qquad\qquad}_{\dot{q}_k}$

$$\sum_{k=1}^f \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

Bsp. I in Kugelkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & m r^2 & 0 \\ 0 & 0 & m r^2 \sin^2 \vartheta \end{pmatrix}$

I in Zylinderkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & m r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

da V unabhängig von $\dot{q}_1, \dots, \dot{q}_f$ ist gilt: $\sum_{k=1}^f \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = 2T$

• Betrachte: $\bar{L} = (q_j, t, \frac{dq_j}{d\tau}, \frac{dt}{d\tau}) = L(q_j, \frac{1}{\frac{dt}{d\tau}} \frac{dq_j}{d\tau}, t) \frac{dt}{d\tau}$

also $t \rightarrow$ q-artige Variable: $q_j = q_j(\tau)$, $t = t(\tau)$, $\dot{t} = \frac{dt}{d\tau} = \dot{q}_{t+1}$

$\bar{L}$ sei invariant unter Zeittrafo: $L^s(q, t) = (q, t+s)$

$\Rightarrow$ (i) Hamiltonsches Prinzip für $\bar{L}$

$$0 = \int_{\tau_1}^{\tau_2} d\tau \bar{L} = \int_{t_1}^{t_2} dt L \Leftrightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

(ii) Noether-Theorem für $\bar{L}$:

Integraler Beweis: $I = \sum_{j=1}^{f+1} \frac{\partial \bar{L}}{\partial \dot{q}_j} \left(\frac{d}{ds} h^s(q_1, \dots, q_f, \dot{q}_{f+1}) \right) \Big|_{s=0} = \frac{\partial \bar{L}}{\partial \dot{q}_{f+1}}$

$$\begin{aligned} &= \frac{\partial \bar{L}}{\partial \dot{q}_{f+1}} = L + \sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \frac{\partial}{\partial \left(\frac{dt}{d\tau} \right)} \left(\frac{1}{\frac{dt}{d\tau}} \right) \frac{dt}{d\tau} \\ &= - \frac{1}{\left(\frac{dt}{d\tau} \right)^2} \frac{\partial L}{\partial \tau} \end{aligned}$$

$$= L - \underbrace{\sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j}_{=2T} = T - V - 2T = -(T+V)$$

4.3 Rotational invariance

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Rotation w.r.t. z axis: $\underline{h}^S: \underline{r}_i \rightarrow \underline{r}'_i = \underline{h}^S(\underline{r}_i) + S \frac{d}{dS} \underline{h}^S(\underline{r}_i) \Big|_{S=0}$
(infinitesimal)

$$= \underline{r}_i + S (\underline{r}_i \times \underline{e}_z)$$

- invariant $L(\underline{r}_i, \dot{\underline{r}}_i) = L(\underline{r}'_i, \dot{\underline{r}}'_i)$ (rotation is an orthogonal transformation $\Rightarrow$ only distance dependent)

$$\Rightarrow \left(\frac{dL}{dS} \right)_{S=0} = - \left(\frac{dL}{dS} \right)_{S=0} = - \sum_{i=1}^N \underline{p}_i \cdot \underline{V}_i \cdot \frac{d\underline{r}'_i}{dS} \Big|_{S=0} = - \underline{e}_z \cdot \sum_{i=1}^N \underline{r}_i \times \underline{F}_i = 0$$

$\underbrace{\quad}_{\text{total torque}}$

Integral of motion

$$\Rightarrow I = \sum_{i=1}^N \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \left(\frac{d\underline{h}^S}{dS} \right) \Big|_{S=0} = \dots = - \underline{e}_z \cdot \sum_{i=1}^N (\underline{r}_i \times \underline{u}_i) = -L_z$$

rotational invariance $\Rightarrow$ conservation of angular momentum

$$\cdot \frac{\partial L}{\partial \varphi} = 0 \quad \text{cyclic variable} \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) = 0 \Rightarrow P_\varphi = \text{const}$$

P_φ

4.4 Temporally translational invariance

(i) no explicitly time-dependent constraint: $\underline{r}_i = \underline{r}_i(q_1, \dots, q_f)$

$$\frac{\partial \underline{r}_i}{\partial t} = 0 \Rightarrow \dot{\underline{r}}_i = \sum_{j=1}^f \frac{\partial \underline{r}_i}{\partial q_j} \dot{q}_j$$

(ii) $\frac{\partial L}{\partial t} = 0$ (scleronomic constraints)

$$\frac{d}{dt} L = \dots = 2 \frac{dT}{dt} \Rightarrow 0 = \frac{d}{dt} (2T - L) = \frac{d}{dt} (T + V)$$

$L = T(\dot{q}) - V(q)$

$$\Rightarrow T + V = \text{const}$$

temporally translational invariance $\Rightarrow$ conservation of energy

Zeitliche Translationsinvarianz:

(i) Zwangsbedingungen nicht explizit t -abhängig

$$\Rightarrow r_i = r_i(q_1, \dots, q_f), \quad \frac{\partial r_i}{\partial t} = 0 \Rightarrow \dot{r}_i = \sum_{j=1}^f \frac{\partial r_i}{\partial q_j} \dot{q}_j$$

$$(ii) \frac{\partial L}{\partial t} = 0$$

• Beobachtung: $\sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j \stackrel{!}{=} \sum_{j=1}^f \left(\frac{\partial T}{\partial \dot{q}_j} \right) \dot{q}_j = \left(\sum_{j=1}^f m_j \dot{q}_j \dot{q}_j \right) \stackrel{!}{=} 2T \quad (*)$

$L = T(\dot{q}) - V(q)$
 $\frac{1}{2} \sum_{j=1}^f m_j \dot{q}_j^2$
 Achtung! (Euler-Satz)
 genau! Siehe 3.2a
 gilt nicht immer!

$$\frac{d}{dt} L = \sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial q_j} \dot{q}_j \right) + \frac{\partial L}{\partial t}$$

$$= \sum_{j=1}^f \left(\frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j \right)$$

$$= \frac{d}{dt} \left(\sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) \stackrel{(*)}{=} 2 \frac{dT}{dt}$$

$$\Rightarrow 0 = \frac{d}{dt} (2T - L) \stackrel{L=T-V}{=} \frac{d}{dt} (T+V) \Rightarrow T+V = \text{const.}$$

Zeittranslationsinvarianz $\Rightarrow$ Energieerhaltung

Stetigkeitsbedingung $\frac{\partial L}{\partial t} = 0$
 Zwangsbedingung

$$\Rightarrow E = T+V = \text{const.}$$

Kinetische Energie: Nachtrag

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} \sum_{j,k} T_{jk} \dot{q}_j \dot{q}_k$$

mit $T_{jk} = \sum_{i=1}^N m_i \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} \right) \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_k} \right)$ abhängig von $q_1, \dots, q_k$!

T ist homogen quadratische Funktion in $\dot{q}_1, \dots, \dot{q}_f$:

$$T(\lambda \dot{q}_1, \dots, \lambda \dot{q}_f) = \lambda^2 T(\dot{q}_1, \dots, \dot{q}_f)$$

$$\frac{\partial}{\partial \lambda} \Big|_{\lambda=1} \Rightarrow \sum_k \frac{\partial T}{\partial (\lambda \dot{q}_k)} \frac{\partial (\lambda \dot{q}_k)}{\partial \lambda} \Big|_{\lambda=1} = 2 \lambda T(\dot{q}_1, \dots, \dot{q}_f) \Big|_{\lambda=1}$$

$\underbrace{\qquad\qquad\qquad}_{\dot{q}_k}$

$$\sum_{k=1}^f \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

Bsp. I in Kugelkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \sin^2 \vartheta \end{pmatrix}$

I in Zylinderkoordinaten $\{T_{jk}\} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

da V unabhängig von $\dot{q}_1, \dots, \dot{q}_f$ ist gilt: $\sum_{k=1}^f \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = 2T$

• Betrachte: $\bar{L} = (q_j, t, \frac{dq_j}{d\tau}, \frac{dt}{d\tau}) = L(q_j, \frac{1}{\frac{dt}{d\tau}} \frac{dq_j}{d\tau}, t) \frac{dt}{d\tau}$

also $t \rightarrow$ q-artige Variable: $q_j = q_j(\tau)$, $t = t(\tau)$, $\dot{t} = \frac{dt}{d\tau} = \dot{q}_{t+1}$

$\bar{L}$ sei invariant unter Zeittrafo: $L^s(q, t) = (q, t+s)$

$\Rightarrow$ (i) Hamiltonsches Prinzip für $\bar{L}$

$$0 = \int_{\tau_1}^{\tau_2} d\tau \bar{L} = \int_{t_1}^{t_2} dt L \Leftrightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

(ii) Noether-Theorem für $\bar{L}$:

Integraler Beweis: $I = \sum_{j=1}^{f+1} \frac{\partial \bar{L}}{\partial \dot{q}_j} \left(\frac{d}{ds} h^s(q_1, \dots, q_f, \dot{q}_{f+1}) \right) \bigg|_{s=0} = \frac{\partial \bar{L}}{\partial \dot{q}_{f+1}}$

$$\begin{aligned} &= \frac{\partial \bar{L}}{\partial \left(\frac{dt}{d\tau} \right)} = L + \sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \frac{\partial}{\partial \left(\frac{dt}{d\tau} \right)} \left(\frac{1}{\frac{dt}{d\tau}} \frac{dq_j}{d\tau} \right) \frac{dt}{d\tau} \\ &= - \frac{1}{\left(\frac{dt}{d\tau} \right)^2} \frac{\partial L}{\partial \dot{q}_j} \end{aligned}$$

$$= L - \underbrace{\sum_{j=1}^f \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j}_{=2T} = T - V - 2T = -(T+V)$$

4.5 Das Zweikörperproblem

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Idee: Herleitet die Bewegungsgleichung über Reduzierung der Freiheitsgrade / Identifizierung der Integrale der Bewegung:

Allgemein: f Freigrade $\Rightarrow f$ DGL 2. Ordnung $\Rightarrow 2f$ Integrationskonstanten (Anfangsbedingungen)
 $\Rightarrow$ max. $2f$ Integrale der Bewegung
 $\uparrow$
(dann wäre Problem gelöst: $I_r(\underline{x}, \underline{\dot{x}}) = C_r, r=1, \dots, f$)

Hier: **Zweikörperproblem**: 2 Massen m_1, m_2 mit innerer Wechselwirkung

$V(|\underline{x}_1 - \underline{x}_2|)$: **Zentralpotential**

(Sonne-Erde, Erde-Mond)

$\Rightarrow f = 6 \Rightarrow 12$ Integrale der Bewegung möglich

(i) $V(|\underline{x}_1 - \underline{x}_2|)$ translationsinvariant

$$\Rightarrow \underline{P} = \underline{P}_1 + \underline{P}_2 = \text{const} \quad (\text{Impuls } \underline{P} = M \underline{\dot{R}})$$

$$\Rightarrow \text{Schwerpunkt: } \underline{R} = \frac{1}{M} \underline{P} t + \underline{R}_0, \quad M = m_1 + m_2$$

$\Rightarrow$ Schwerpunkt bewegt sich geradlinig und gleichförmig

$\Rightarrow 6$ Integrationskonstanten $\underline{P}, \underline{R}_0$

(ii) $V(|\underline{x}_1 - \underline{x}_2|)$ rotationsinvariant

$$\Rightarrow \underline{L} = m_1 \underline{r}_1 \times \underline{\dot{r}}_1 + m_2 \underline{r}_2 \times \underline{\dot{r}}_2 = \text{const} \quad (\text{Drehimpuls})$$

$\Rightarrow 3$ Integrationskonstanten $\underline{L}$

(iii) zeitliche Translationsinvariant, konservativer Kraft

$$\Rightarrow \underline{E} = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 + V(|\underline{x}_1 - \underline{x}_2|) = \text{const} \quad (\text{Energie})$$

$\Rightarrow 1$ Integrationskonstante $\underline{E}$

$\Rightarrow 10$ Integrale der Bewegung identifiziert

$\Rightarrow$ Es bleiben $12 - 10 = 2$ Integrationskonstanten (Nullpunkt von Zeit- & Winkel skala)

4.5.1 Impuls- & Drehimpulserhaltung

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Idee: Lagrange-Formalismus: $L = T - V = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 - V(|\underline{r}_1 - \underline{r}_2|)$

Verallgemeinerte Koordinaten:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underline{R} = \frac{1}{M} (m_1 \underline{r}_1 + m_2 \underline{r}_2)$$

Schwerpunkts Koordinate

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \end{pmatrix} = \underline{r} = \underline{r}_1 - \underline{r}_2$$

Relative Koordinate

$$\hookrightarrow \underline{r}_1 = \underline{R} + \frac{m_2}{M} \underline{r}, \quad \underline{r}_2 = \underline{R} - \frac{m_1}{M} \underline{r}$$

$$\dot{\underline{r}}_1 = \dot{\underline{R}} + \frac{m_2}{M} \dot{\underline{r}}, \quad \dot{\underline{r}}_2 = \dot{\underline{R}} - \frac{m_1}{M} \dot{\underline{r}}$$

$$\Rightarrow L = \frac{M}{2} \dot{\underline{R}}^2 + \frac{1}{2} \mu \dot{\underline{r}}^2 - V(r), \quad r = |\underline{r}| \text{ Abstand}$$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ reduzierte Masse

• $\frac{\partial L}{\partial \underline{R}_K} = 0 \Rightarrow \underline{R}$ ist zyklische Variable $\Rightarrow \frac{\partial L}{\partial \dot{\underline{R}}_K} = M \dot{\underline{R}}_K = P_K = \text{const}$

$$\Rightarrow \underline{R} = \frac{1}{M} \underline{P} + \underline{R}_0$$

• Schwerpunktssystem als Inertialsystem! Bedd. $\underline{R} = \dot{\underline{R}} = 0$

$$\Rightarrow L = \frac{1}{2} \mu \dot{\underline{r}}^2 - V(r), \quad \underline{r}_1 = \frac{m_2}{M} \underline{r}, \quad \underline{r}_2 = -\frac{m_1}{M} \underline{r}$$

$$\dot{\underline{r}}_1 = \frac{m_2}{M} \dot{\underline{r}}, \quad \dot{\underline{r}}_2 = -\frac{m_1}{M} \dot{\underline{r}}$$

Drehimpuls: $\underline{L} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2 = \frac{m_1 m_2}{M}$

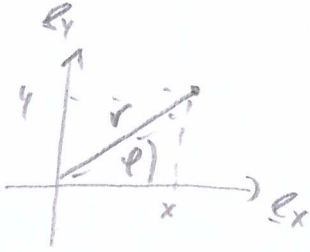
$$= m_1 \frac{m_2}{M} \underline{r} \times \frac{m_2}{M} \dot{\underline{r}} + m_2 \left(-\frac{m_1}{M}\right) \underline{r} \times \left(-\frac{m_1}{M}\right) \dot{\underline{r}}$$

$$= \left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2} \right) \underline{r} \times \dot{\underline{r}} = \mu \underline{r} \times \dot{\underline{r}} = \text{const}$$

$$\frac{(m_1 + m_2) m_1 m_2}{M^2} = \frac{m_1 m_2}{M} = \mu$$

$\Rightarrow \underline{L} \cdot \underline{r} = 0, \underline{L} \cdot \dot{\underline{r}} = 0 \Rightarrow \underline{r}$ und $\dot{\underline{r}}$ in Ebene senkrecht zu $\underline{L}$ (im Schwerpunktssystem)

→ 2D Bewegung! ⇒ Polarkoordinaten
(Objekt in $\mathbb{R}_2$)



$$x = r \cos \varphi \Rightarrow \dot{x} = \dot{r} \cos \varphi - r \dot{\varphi} \sin \varphi$$

$$y = r \sin \varphi \Rightarrow \dot{y} = \dot{r} \sin \varphi + r \dot{\varphi} \cos \varphi$$

$$\Rightarrow \dot{r}^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 (\cos^2 \varphi + \sin^2 \varphi) + r^2 \dot{\varphi}^2 (\sin^2 \varphi + \cos^2 \varphi) \\ = \dot{r}^2 + r^2 \dot{\varphi}^2$$

$$\Rightarrow L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$$

$$L_x = L_y = 0$$

$$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi \text{ ist zyklisch} \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = l_z = \frac{1}{2} l = \text{const}$$

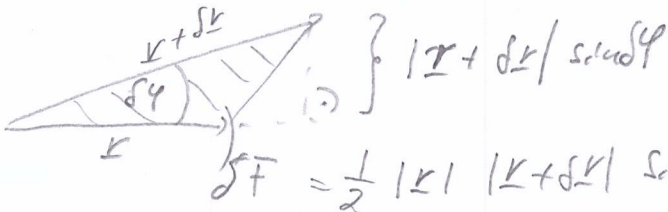
$$\uparrow$$

$$m(x\dot{y} - y\dot{x}) = m r^2 \dot{\varphi}$$

• **2. Keplers Gesetz:** Flächengesetz (geometrische Interpretation von $m r^2 \dot{\varphi} = l = \text{const}$)

Der Radiusvektor $\underline{r}$ überstreicht in gleichen Zeiten gleiche Flächen.

(Flächengeschwindigkeit $\frac{dT}{dt} = \text{const}$)



$$dF = \frac{1}{2} |r| |r + \delta r| \sin \delta \varphi \approx \frac{1}{2} |r| |r| \delta \varphi$$

$$\delta r, \delta \varphi \rightarrow 0$$

$$\sin \delta \varphi \approx \delta \varphi$$

$$\Rightarrow \frac{dT}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2} r^2 \dot{\varphi} = \frac{l}{2m} = \text{const}!$$

$$m r^2 \dot{\varphi} = l$$

4.5 Two-body problem

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2 (point) masses in 3D $\Rightarrow f = 6$ degrees of freedom

$\Rightarrow 2f = 12$ independent constants / initial values

potential: $V(\underline{r}_1, \underline{r}_2) = V(|\underline{r}_1 - \underline{r}_2|)$ distance-dependent

(i) translational invariance $\Rightarrow$ conservation of momentum $\underline{P} = M \dot{\underline{R}} = \text{const}$

$$\Rightarrow \underline{R}(t) = \frac{\underline{P}}{M} t + \underline{R}_0$$

$\Rightarrow 3$ initial values $\underline{P}$

3 " " $\underline{R}_0$

(ii) rotational invariance $\Rightarrow$ conservation of angular momentum $\underline{L} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2 = \text{const}$

$\Rightarrow 3$ initial conditions $\underline{L}$

(iii) temporally translational invariance $\Rightarrow$ conservation of energy

$$E = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 + V(|\underline{r}_1 - \underline{r}_2|) = \text{const}$$

$\Rightarrow 1$ initial value E

$\Rightarrow 10$ of 12 initial values given by system setup

$$\left. \begin{array}{l} \text{coordinates of center of mass: } \underline{R} = \frac{1}{M} (m_1 \underline{r}_1 + m_2 \underline{r}_2) \\ \text{relative coordinate: } \underline{r} = \underline{r}_1 - \underline{r}_2 \end{array} \right\} \Rightarrow \begin{array}{l} \underline{r}_1 = \underline{R} + \frac{m_2}{M} \underline{r} \\ \underline{r}_2 = \underline{R} - \frac{m_1}{M} \underline{r} \end{array}$$

$$\Rightarrow \text{Lagrangian: } L = \frac{M}{2} \dot{\underline{R}}^2 + \frac{m}{2} \dot{\underline{r}}^2 - V(r), \quad m = \frac{m_1 m_2}{m_1 + m_2} \text{ reduced mass}$$

$$\Rightarrow \underline{R} \text{ is cyclic variable } \Rightarrow \frac{\partial L}{\partial \dot{\underline{R}}} = \text{const} = \underline{P}_R \Rightarrow \underline{R}(t) = \frac{\underline{P}}{M} t + \underline{R}_0$$

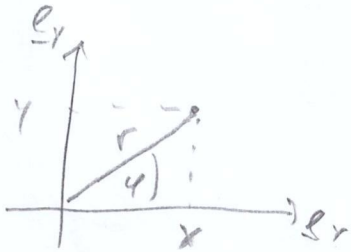
$$\text{w.l.g. } \underline{P} = 0 = \underline{P}_R \Rightarrow \underline{R} = 0$$

angular momentum $\underline{l} = \dots = m \underline{r} \times \dot{\underline{r}}$

$$\Rightarrow \underline{l} \cdot \underline{r} = 0, \quad \underline{l} \cdot \dot{\underline{r}} = 0 \quad (\text{motion in a plane})$$

$\Rightarrow$ effective 2D problem (w.l.g. $\underline{l} \parallel \underline{e}_z$)

$\Rightarrow$ polar coordinates

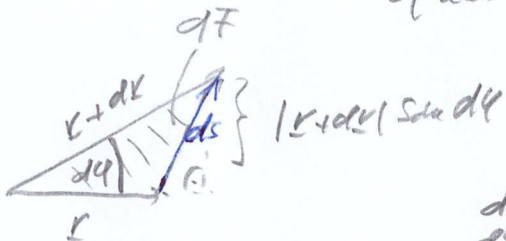


$$\left. \begin{aligned} x &= r \cos \varphi \Rightarrow \dot{x} = \dot{r} \cos \varphi - r \dot{\varphi} \sin \varphi \\ y &= r \sin \varphi \Rightarrow \dot{y} = \dot{r} \sin \varphi + r \dot{\varphi} \cos \varphi \end{aligned} \right\} \Rightarrow \dot{\underline{r}}^2 = \dot{r}^2 + r^2 \dot{\varphi}^2$$

$$\Rightarrow L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r) \quad m(x\dot{y} - \dot{x}y)$$

$$\Rightarrow \varphi \text{ is a cyclic variable} \quad \frac{\partial L}{\partial \varphi} = 0 \quad \frac{\partial L}{\partial \dot{\varphi}} = \text{const} = P_\varphi = m r^2 \dot{\varphi} = \underline{l}_z = \underline{l} \parallel \underline{e}_z$$

$\Rightarrow$ Kepler's 2nd law: A line joining a planet and the sun sweeps out equal areas during equal intervals of time.



$$d\bar{r} = \frac{1}{2} |\underline{r}| |\underline{r}| \sin \varphi d\varphi$$

$$\frac{dr}{dt} > 0 \quad \frac{d\varphi}{dt} > 0 \quad \frac{d\bar{r}}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2} r^2 \frac{l}{m r^2} = \frac{l}{2m} = \text{const}$$

$$\begin{aligned} \text{alternative: } d\bar{r} &= \frac{1}{2} |\underline{r} \times d\underline{r}| = \frac{1}{2} |\underline{r} \times \dot{\underline{r}} dt| \Rightarrow \frac{d\bar{r}}{dt} = \frac{1}{2} |\underline{r} \times \dot{\underline{r}}| \\ &= \frac{1}{2m} |\underline{r} \times m \dot{\underline{r}}| \\ &= \frac{1}{2m} \underline{l} \end{aligned}$$

4.5.2 Energieerhaltung und Bahngleichung

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Lagrange-Gleichung 2. Art für r : $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}}{\partial r} = 0$

$\uparrow$ $\quad \quad \quad \uparrow$
 $m \ddot{r}$ $\quad \quad \quad m r \dot{\varphi}^2 - V'(r)$

$$\Rightarrow m \ddot{r} - \underbrace{m r \dot{\varphi}^2}_{\text{Zentrifugalkraft}} + V'(r) = 0 \quad \left. \vphantom{\ddot{r}} \right\} \dot{\varphi} = \frac{L}{m r^2}$$

$$\Rightarrow m \ddot{r} - \frac{L^2}{m r^3} + V'(r) = 0$$

$\uparrow$
 $\dot{\varphi} = \frac{L}{m r^2}$

$\downarrow \cdot \dot{r}$

$$\underbrace{m \dot{r} \ddot{r}}_{\frac{d}{dt} \left(\frac{m}{2} \dot{r}^2 \right)} + \frac{L^2}{m} \underbrace{\left(-\frac{\dot{r}}{r^3} \right)}_{\frac{d}{dt} \frac{1}{2r^2}} + \underbrace{\dot{r} V'(r)}_{\frac{d}{dt} V(r)}$$

$$\Rightarrow \frac{d}{dt} \left(\frac{m}{2} \dot{r}^2 + \frac{L^2}{2m r^2} + V(r) \right) = 0 \quad \int dt \quad E = T + V$$

$$\frac{m}{2} \dot{r}^2 + \frac{L^2}{2m r^2} + V(r) = E = \text{const}$$

$$\underbrace{\frac{m}{2} \dot{r}^2 + \frac{m r^2 \dot{\varphi}^2}{2}}_{\frac{m}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2)} = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) = T$$

Alternativ: $E = \frac{m}{2} \dot{r}^2 + \tilde{V}(r)$ mit **effektivem Radialpotential**

$$\tilde{V}(r) = V(r) + \frac{L^2}{2m r^2}$$

$$\Rightarrow \dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m} (E - \tilde{V}(r))}$$

Trennung der Variable $\int_{t_0}^t dt' = \int_{r_0}^r \frac{dr'}{\sqrt{\frac{2}{m} (E - \tilde{V}(r'))}}$ $\Rightarrow t = t(r) \Rightarrow \text{Umkehrung: } r = r(t)$

$$\Rightarrow \varphi(t) \text{ aus } \dot{\varphi} = \frac{d\varphi}{dt} = \frac{L}{m r^2(t)} \quad \int_{\varphi_0}^{\varphi} d\varphi' = \int_{t_0}^t dt' \frac{L}{m r^2(t')} \Rightarrow \varphi = \varphi(t)$$

Bewegungsgleichung: $r = r(\varphi)$, $\varphi = \varphi(r)$

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$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \frac{m r^2}{l} \sqrt{\frac{2}{m} (E - \tilde{V}(r))}$$

$$\int d\varphi = \int \frac{dr}{r^2 \sqrt{\frac{2m}{l^2} (E - \tilde{V}(r))}}$$

4.5.3 Plausibilitätsprüfung & Kepler'sche Gesetze

Nun: Gravitationspotential: $V(r) = - \frac{\gamma m_1 m_2}{r}$

$$\Rightarrow \tilde{V}(r) = - \frac{K}{r} + \frac{l^2}{2mr^2}, \quad K = \gamma m_1 m_2 > 0$$

$$r \rightarrow 0: \tilde{V}(r) \sim \frac{l^2}{2mr^2} \rightarrow \infty$$

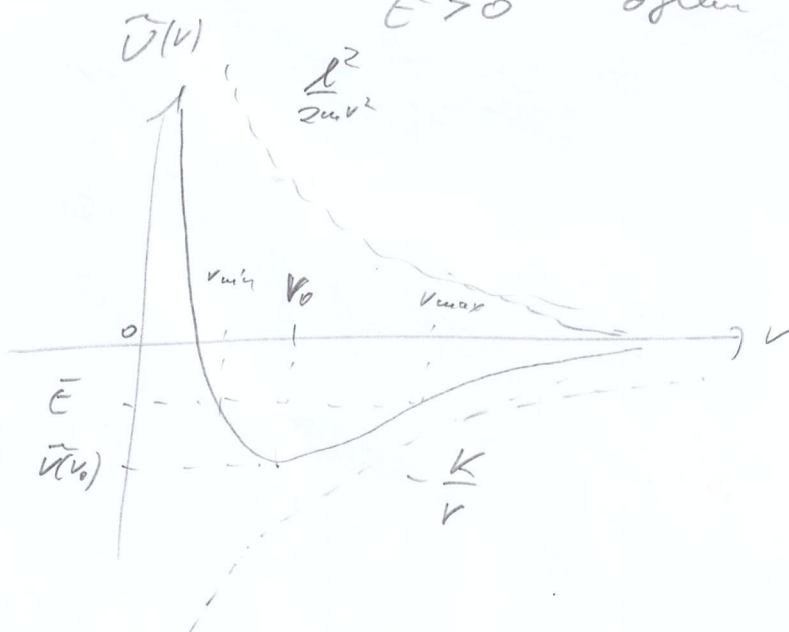
$$r \rightarrow \infty: \tilde{V}(r) \sim - \frac{K}{r} \rightarrow 0$$

$$\text{Minimum: } \frac{d\tilde{V}}{dr} = \frac{K}{r^2} - \frac{l^2}{mr^3} \stackrel{!}{=} 0 \Rightarrow r_0 = \frac{l^2}{mK} \Rightarrow \tilde{V}(r_0) = - \frac{mK^2}{2l^2}$$

Wegen $\frac{m}{2} \dot{r}^2 = E - \tilde{V}(r)$ ist eine Bewegung nur für $\tilde{V}(r) < E$ möglich

$$\Rightarrow - \frac{mK^2}{2l^2} < E < 0 \quad \text{geschlossene Bahn}$$

$$E > 0 \quad \text{offene Bahn}$$



Für $-\frac{mk^2}{2\ell^2} < E < 0$ sind die Wendepunkte durch

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$$\tilde{V}(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} = E \quad \text{besitzt (E < 0)}$$

$$\Rightarrow V_{\text{min/max}} = \frac{1}{2|E|} \left(k \pm \sqrt{k^2 - \frac{2\ell^2|E|}{m}} \right)$$

$$Er^2 + kr - \frac{\ell^2}{2m} = 0$$

$$r^2 - \frac{k}{|E|}r + \frac{\ell^2}{2m|E|} = 0$$

$$\hookrightarrow V_{\text{min/max}} = \frac{1}{2|E|} \left(k \mp \sqrt{k^2 - 4 \frac{\ell^2}{2m}} \right)$$

$$= \frac{1}{2|E|} \left(k \mp \sqrt{k^2 - \frac{2\ell^2|E|}{m}} \right)$$

$$E > 0: \text{ nur 1 Wendepunkt: } V_{\text{min}} = \frac{1}{2E} \left(\sqrt{k^2 + \frac{2\ell^2 E}{m}} - k \right)$$

$$\text{Bohrerfordnung: } \varphi - \varphi_0 = \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2} (E \tilde{V}(r'))}}$$

$$= \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2} E + \frac{2mk}{\ell^2 r'} - \frac{1}{r'^2}}}$$

$$= \dots = \int_{r_0}^r dr' \quad \text{mit} \quad \cos \vartheta = \frac{1}{\sqrt{D}} \left(\frac{1}{r'} - \frac{mk}{\ell^2} \right)$$

4.5.2 Conservation of energy & trajectory (around the ^{Star} sun)

Euler-Lagrange equation: $\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$$

$$\Rightarrow m \ddot{r} - m r \dot{\varphi}^2 + \frac{\partial V}{\partial r} = 0, \quad \dot{\varphi} = \frac{l}{m r^2}$$

$$\Rightarrow \dots \Rightarrow E = \frac{m}{2} \dot{r}^2 + \underbrace{\frac{l^2}{2 m r^2}}_{=: \tilde{V}(r)} + V(r) = \text{const} \quad \left| \begin{array}{l} \text{2nd degree polynomial} \\ \frac{l^2}{2 m r^2} \Rightarrow F_c = \frac{l^2}{2 m r^3} \frac{1}{r} \end{array} \right.$$

$$\Rightarrow t(r) - t_0 = \int_{r_0}^r \frac{dr'}{\sqrt{\frac{2}{m} (E - \tilde{V}(r))}} \Rightarrow t(r) \text{ or } r(t)$$

$$\varphi(t) - \varphi_0 = \int_{t_0}^t dt' \frac{l}{m r^2(t')} \rightarrow \varphi(t)$$

$$\frac{dr}{dt} = \dot{r} = \frac{m \dot{r}^2}{l} \sqrt{\frac{2}{m} (E - \tilde{V}(r))}$$

$$\Rightarrow \varphi(r) - \varphi_0 = \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2}{m} (E - \tilde{V}(r))}} \Rightarrow \varphi(r), r(\varphi)$$

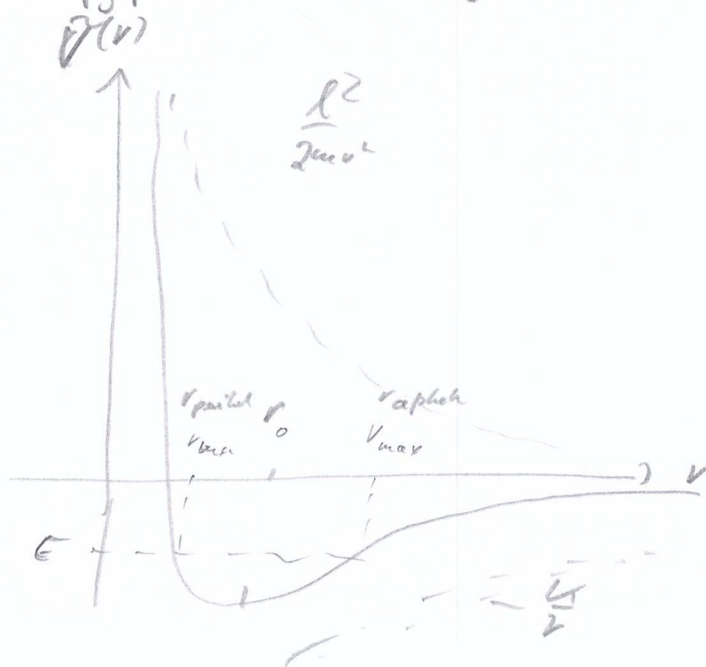
4.5.3 motion of planet & Kepler's laws

now: $V(r) = - \frac{\gamma m_1 m_2}{r} = - \frac{K}{r} \Rightarrow \tilde{V}(r) = - \frac{K}{r} + \frac{l^2}{2 m r^2} \quad \left\{ \begin{array}{l} r \rightarrow 0 \Rightarrow \frac{l^2}{2 m r^2} \rightarrow \infty \\ r \rightarrow \infty \Rightarrow - \frac{K}{r} \rightarrow 0 \end{array} \right.$

motion possible only for $E > \tilde{V}(r)$ ($\frac{1}{2} m \dot{r}^2 = E - \tilde{V}$)

closed orbit $-\frac{m K^2}{2 l^2} < E < 0$

diverging path $E > 0$



$$r_{\min/\max} = \frac{1}{2|E|} \left(4 \pm \sqrt{4^2 - \frac{2 l^2 |E|}{m}} \right)$$

$$\varphi - \varphi_0 = \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2}{m} E + \frac{2 K}{r'} + \frac{1}{r'^2}}}$$

Lösen des Integrals:

$$\varphi - \varphi_0 = \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2} (E - \tilde{V}(r'))}} = \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2} \left(E + \frac{k}{r'} - \frac{\ell^2}{2mr'^2} \right)}}$$

Benutze (i) quadratische Ergänzung und (ii) Substitution

$$\begin{aligned} \text{(i)} \quad \frac{2m}{\ell^2} (E - \tilde{V}(r)) &= \frac{2mE}{\ell^2} + \frac{2mk}{\ell^2 r'} - \frac{1}{r'^2} = - \left(\frac{1}{r'} - \frac{mk}{\ell^2} \right)^2 + \frac{m^2 k^2}{\ell^4} + \frac{2mE}{\ell^2} \\ &= D - \left(\frac{1}{r'} - \frac{mk}{\ell^2} \right)^2 \quad \text{mit } D = \frac{2m}{\ell^2} \left(\frac{mk^2}{2\ell^2} + E \right) > 0 \\ &= D \left[1 - \frac{1}{D} \left(\frac{1}{r'} - \frac{mk}{\ell^2} \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{(ii) Substitution: } \cos \varphi' &= \frac{1}{\sqrt{D}} \left(\frac{1}{r'} - \frac{mk}{\ell^2} \right) \Rightarrow -\sin \varphi' d\varphi' = -\frac{dr'}{\sqrt{D} r'^2} \\ &\quad \frac{d}{dr'} (\cos \varphi') = -\frac{\sin \varphi'}{r'^2} \dots \\ \Rightarrow \varphi - \varphi_0 &= \int_{r_0}^r \frac{dr'}{r'^2} \frac{1}{\sqrt{D}} \frac{1}{\sqrt{1 - \frac{1}{D} \left(\frac{1}{r'} - \frac{mk}{\ell^2} \right)^2}} = \int_{\varphi_0}^{\varphi} d\varphi' \frac{1}{\sin \varphi'} \frac{1}{\sqrt{1 - \cos^2 \varphi'}} = \int_{\varphi_0}^{\varphi} d\varphi' \\ &= \varphi - \varphi_0 = \arccos \left[\frac{1}{\sqrt{D}} \left(\frac{1}{r} - \frac{mk}{\ell^2} \right) \right] - \arccos \left[\frac{1}{\sqrt{D}} \left(\frac{1}{r_0} - \frac{mk}{\ell^2} \right) \right] \end{aligned}$$

r_0, φ_0 Integrationskonstanten!

$$\text{denn: } \varphi_0 = \arccos \left[\frac{1}{\sqrt{D}} \left(\frac{1}{r_0} - \frac{mk}{\ell^2} \right) \right] \Rightarrow \varphi(r) = \arccos \left[\frac{1}{\sqrt{D}} \left(\frac{1}{r} - \frac{mk}{\ell^2} \right) \right]$$

$$\begin{aligned} \Rightarrow \frac{1}{r(\varphi)} &= \frac{mk}{\ell^2} + \sqrt{D} \cos \varphi = \frac{mk}{\ell^2} \left(1 + \epsilon \cos \varphi \right) \\ &\quad \uparrow \\ &= \sqrt{D} \frac{\ell^2}{mk} = \sqrt{1 + \frac{2E\ell^2}{mk^2}} \geq 0 \end{aligned}$$

$$\Rightarrow r(\varphi) = \frac{\frac{\ell^2}{mk}}{1 + \epsilon \cos \varphi} \quad \begin{cases} \epsilon = 0: r(\varphi) = \frac{\ell^2}{mk} = \text{const} \Rightarrow \text{Kreis} \\ 0 < \epsilon < 1: \text{Ellipse} \end{cases}$$

3. Keplers Gesetze

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$$T^2 \sim a^3 \quad (T: \text{Umlaufzeit}, a: \text{große Halbachse})$$

Ellipsenfläche $F = \pi a b$

mit $\frac{dF}{dt} = \frac{L}{2m} = \text{const}$ folgt: $F = \pi a b \int_0^T \frac{dF}{dt} dt = \frac{L}{2m} T$

$$\Rightarrow T = \frac{2m\pi ab}{L} \stackrel{\uparrow}{=} \frac{2m\pi a}{L} \frac{L}{\sqrt{m\mu}} \sqrt{a} = 2\pi \sqrt{\frac{m}{\mu}} a^{3/2}$$

$$\frac{b^2}{a} = \frac{L^2}{m\mu} \Rightarrow b = \frac{L}{\sqrt{m\mu}} \sqrt{a}$$

$$\Rightarrow T^2 \sim a^3$$

mit $\mu = \frac{m_1 m_2}{m_1 + m_2}$, $k = \gamma m_1 m_2$ folgt: $\frac{m}{\mu} = \frac{1}{\gamma(m_1 + m_2)}$

Für m_1 (Planet) $\ll m_2$ (Sonne): $\frac{m}{\mu} \approx \frac{1}{\gamma m_2} \Rightarrow T \approx 2\pi \sqrt{\frac{a^3}{\gamma m_2}}$

4.5.3 motion of planets & Kepler's laws

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$$\text{Solve the integral } \varphi - \varphi_0 = \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2m}{l^2} (E - \tilde{V}(r'))}}$$

$$= \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{\frac{2m}{l^2} \left(E + \frac{k}{r'} - \frac{l^2}{2mr'^2} \right)}}$$

by (i) quadratic expansion and (ii) substitution

$$(i) \frac{2m}{l^2} (E - \tilde{V}(r)) = \dots = D \left[1 - \frac{1}{D} \left(\frac{1}{r'} - \frac{mk}{l^2} \right)^2 \right] \text{ with } D = \frac{2m}{l^2} \left(\frac{mk^2}{2l^2} + E \right)$$

$$(ii) \cos \vartheta' = \frac{1}{\sqrt{D}} \left(\frac{1}{r'} - \frac{mk}{l^2} \right) \Rightarrow \frac{d}{dr'} \cos \vartheta' =$$

$$\Rightarrow \frac{d}{dr'} \cos \vartheta' = \frac{d}{dr'} \frac{1}{\sqrt{D}} \left(\frac{1}{r'} - \frac{mk}{l^2} \right)$$

$$\Rightarrow -\sin \vartheta' \frac{d\vartheta'}{dr'} = -\frac{1}{\sqrt{D} r'^2} \Rightarrow \frac{dr'}{\sqrt{D} r'^2} = + \sin \vartheta' d\vartheta'$$

$$\Rightarrow \varphi - \varphi_0 = \int_{\vartheta_0}^{\vartheta} \sin \vartheta' \frac{1}{\sqrt{1 - \cos^2 \vartheta'}} d\vartheta' = \int_{\vartheta_0}^{\vartheta} d\vartheta' = \vartheta - \vartheta_0$$

$$\Rightarrow \varphi(r) = \arccos \left[\frac{1}{\sqrt{D}} \left(\frac{1}{r'} - \frac{mk}{l^2} \right) \right]$$

$$\Rightarrow r(\varphi) = \frac{l^2/mk}{1 + E \cos \varphi}$$

$E = 0$: circle

$0 < E < 1$: ellipse $\left(-\frac{mk^2}{2l^2} < E < 0 \right)$

$E = 1$: parabola

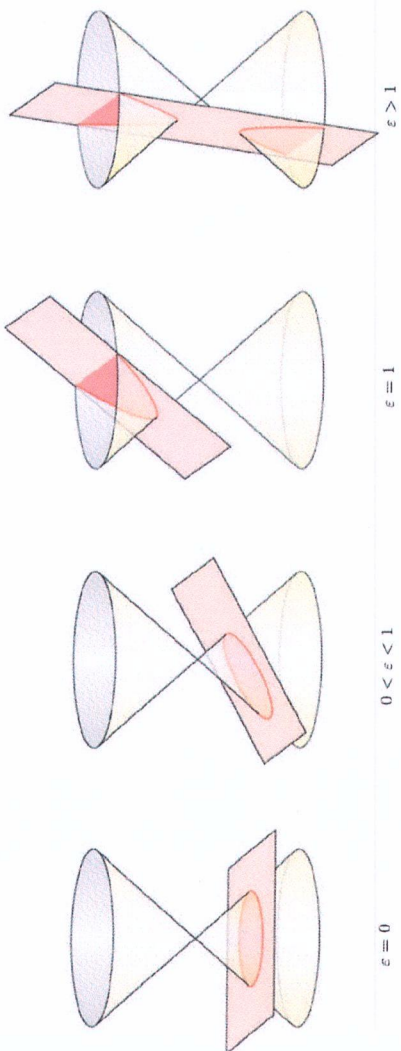
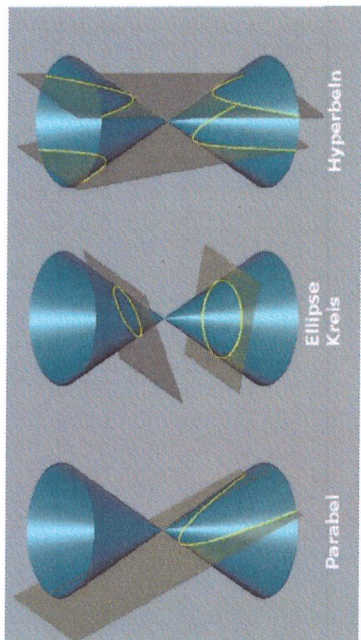
$E > 1$: hyperbola

E = numeric eccentricity

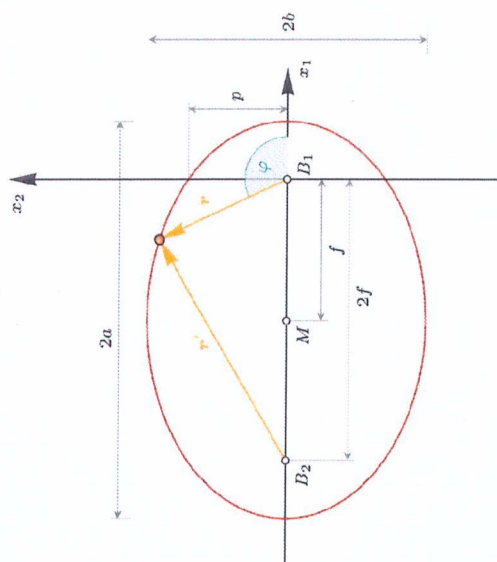
Kepler's 3rd law: $T^2 \sim a^3$

$\uparrow$
period

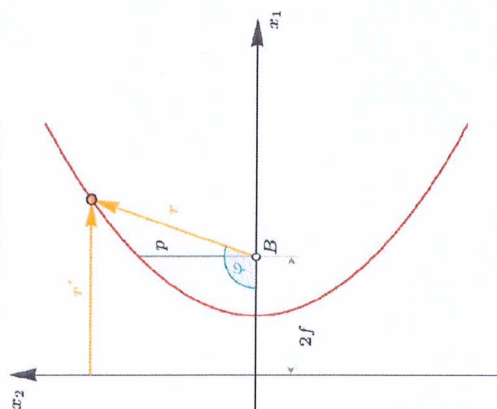
$\uparrow$
semi-axis of ellipse



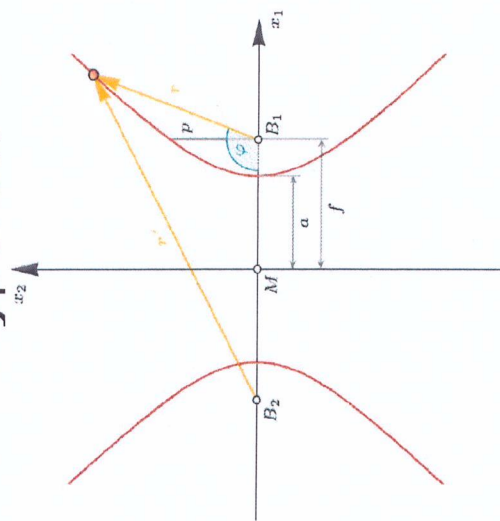
Ellipse



Parabel



Hyperbel



5.1 Legendre-Transform

5.2 Die Hamiltonschen Gleichungen

5.3 Symplektische Struktur des Phasenraums

5.4 Das Liouville'sche Satz

5.5 Poisson-Klammern

Idee: Lagrange-Theorie: generalisierte Koordinaten $q_1, \dots, q_f$
 und Geschwindigkeiten $\dot{q}_1, \dots, \dot{q}_f$
 $\Rightarrow L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t) \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0, k=1, \dots, f$
 f Dgls 2. Ordnung

Hamilton-Formalismus: $(q_k, \dot{q}_k, t) \rightarrow (q_k, p_k, t)$

$$H(q_1, \dots, q_f, p_1, \dots, p_f, t) \Rightarrow \dots \Rightarrow \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = - \frac{\partial H}{\partial q_k}$$

2f Dgls 1. Ordnung

5.1 Legendre Transformation

Idee: Für $y=f(x)$ soll statt x die Variable $\frac{df}{dx} = u$ verwendet werden.
 $f'(u)$

Annahme: $\frac{df}{dx} = u$ umkehrbar (für $x = \varphi(u) \Rightarrow \frac{d^2 f}{dx^2} \neq 0$)

Achtung: $y=f(x)+a$ liefert auch dieselbe Steigung $\frac{df}{dx}$!

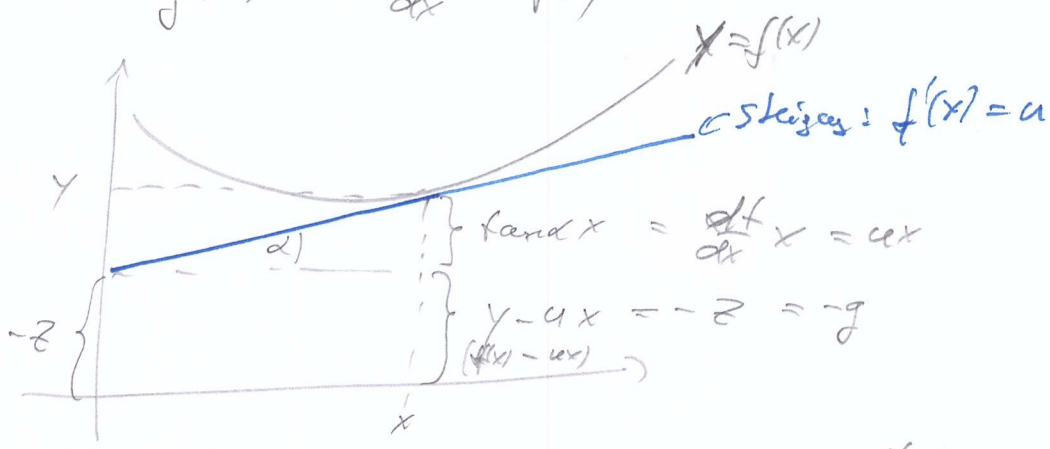


$\Rightarrow$ Legendre-Transform

$$x \rightarrow u = \frac{df}{dx}$$

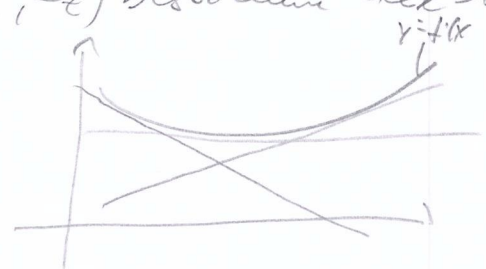
$$y \rightarrow z = x u - \underbrace{f(x)}_y = \varphi(u) u - f(\varphi(u)) \equiv g(u)$$

$$\Rightarrow g(u) = x \frac{df}{dx} - f(x)$$



Also: statt $y = f(x)$: Trafo $x \rightarrow u = \frac{df}{dx}$ und (negativer) Achsenabschnitt der Tangente ($-z$)

$(u, -z)$ bestimmen die Scheitel der Erhaltungss.



Anwendung auf $L(q, \dot{q}, t)$:

$$y = L, x = \dot{q}, u = \frac{\partial L}{\partial \dot{q}} = p \Rightarrow z = \dot{q} p - L = H(q, p, t)$$

Hamilton-Funktion

(ii) $L(q_1, \dots, q_k, \dot{q}_1, \dots, \dot{q}_k, t)$:

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \Rightarrow H(q_1, \dots, q_k, p_1, \dots, p_k, t) = \sum_{j=1}^k \dot{q}_j p_j - L(q_1, \dots, q_k, \dot{q}_1, \dots, \dot{q}_k, t)$$

($\det \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \neq 0$, sonst $p_k = \frac{\partial L}{\partial \dot{q}_k}$ nicht nach $\dot{q}_k$ auflösbar)

Ziel: Bewegungsgleichungen für $q_1, \dots, q_f, p_1, \dots, p_f$:

$$H(q_1, \dots, q_f, p_1, \dots, p_f, t) = \sum_{j=1}^f \dot{q}_j p_j - L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$$

totaler Differential

$$\Rightarrow dH = \sum_{j=1}^f \left\{ \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j \right\} + \frac{\partial H}{\partial t} dt \quad (\text{linke Seite})$$

$$= \sum_{j=1}^f \left\{ \dot{q}_j dp_j + p_j dq_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right\} - \frac{\partial L}{\partial t} dt \quad (\text{rechte Seite})$$

$-dL$

Vergleich LS mit RS liefert:

$$\left. \begin{aligned} dq_j: \quad \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ dp_j: \quad \frac{\partial H}{\partial q_j} &= -\frac{\partial L}{\partial q_j} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = -\frac{d}{dt} p_j = \dot{p}_j \end{aligned} \right\} \quad \begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}$$

$$dt: \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Hamilton'sche Gleichungen

Bei zeitlicher Trennung (offenes Universum) (Schrödingers Zwangsbedingungen:

$$\frac{\partial}{\partial t} r_i(q_1, \dots, q_f) = 0, \quad \frac{\partial}{\partial t} \dot{r}_i = 0)$$

$$\sum_{j=1}^f \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T \Rightarrow H = \sum_{j=1}^f \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = 2T - (T - V) = T + V$$

$$\Rightarrow H = T + V : \text{Gesamtenergie}$$

$$\Rightarrow \text{Energieerhaltung: } \frac{dH}{dt} = \sum_{j=1}^f \left\{ \frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right\} + \frac{\partial H}{\partial t} = 0$$

$= 0$

• rheonome Zwangsbedingungen: $H \neq T + V$

Kochrezept der Hamiltonschen Gleichungen:

1. Auswählen generalisierter Koordinaten: $q_1, \dots, q_f$
2. Transformation $\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_f, t)$ und $\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$
3. Berechnen der Lagrange-Funktion $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$
4. Berechnen der generalisierten Impulse $p_j = \frac{\partial L}{\partial \dot{q}_j}$ und deren Umkehrung $\dot{q}_j = \dot{q}_j(q_1, \dots, q_f, p_1, \dots, p_f, t)$
5. Legendre-Transformation $H(q_1, \dots, q_f, p_1, \dots, p_f, t) = \sum_{j=1}^f \dot{q}_j p_j - L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$
6. Aufstellung der Bewegungsgleichungen ($j = 1, \dots, f$):

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{und} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

7. Lösen der Bewegungsgleichungen

Bsp.: a) 1 Teilchen im 3D-Kartesischen Koordinatensystem

$$(i) \quad \varphi_1 = r, \quad \varphi_2 = \varphi, \quad \varphi_3 = z$$

$$(ii) \quad \begin{aligned} x &= r \cos \varphi & \dot{x} &= \dot{r} \cos \varphi - r \dot{\varphi} \sin \varphi \\ y &= r \sin \varphi & \dot{y} &= \dot{r} \sin \varphi + r \dot{\varphi} \cos \varphi \\ z &= z & \dot{z} &= \dot{z} \end{aligned}$$

$$(iii) \quad T = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2), \quad V = V(r, \varphi, z)$$

$$\Rightarrow L(r, \varphi, z, \dot{r}, \dot{\varphi}, \dot{z}) = T - V$$

$$(iv) \quad p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \Rightarrow \dot{r} = \frac{p_r}{m} \quad \text{Radialimpuls}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{m r^2} \quad z\text{-Komponente des Drehimpulses } (L_z)$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z} \Rightarrow \dot{z} = \frac{p_z}{m} \quad z\text{-Komponente des Impulses}$$

$$(v) \quad H = p_r \dot{r} + p_\varphi \dot{\varphi} + p_z \dot{z} - L = \frac{1}{m} \left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2 \right) - \frac{1}{2m} \left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2 \right) + V(r, \varphi, z)$$

$$= \frac{1}{2m} \left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2 \right) + V(r, \varphi, z)$$

$$(vi) \quad \begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, & \dot{p}_r &= -\frac{\partial H}{\partial r} = \left(+\frac{p_\varphi^2}{m r^3} - \frac{\partial V}{\partial r} \right) \\ \dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m r^2}, & \dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi} = -\frac{\partial V}{\partial \varphi} \\ \dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m}, & \dot{p}_z &= -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z} \end{aligned} \quad \left. \begin{aligned} &\text{Radialpotential:} \\ &\dot{r} = \frac{p_r}{m} - \frac{\partial V}{\partial r} \\ &\dot{p}_r = 0 = p_r \end{aligned} \right\}$$

$$\dot{p}_\varphi = 0 \Rightarrow \dot{p}_\varphi = m \rho^2 \dot{\varphi} = \text{const} \quad \text{Drehimpuls exakt!} \quad 97$$

$$\dot{p}_z = 0 \Rightarrow p_z = \text{const} = 0 \quad (\text{ohne Beugung})$$

(b) 1D harmonischer Oszillator

$$(i) - (iii) \quad L = \frac{m}{2} (\dot{q}^2 - \omega^2 q^2)$$

$$(iv) \quad p = \frac{\partial L}{\partial \dot{q}} = m \dot{q}$$

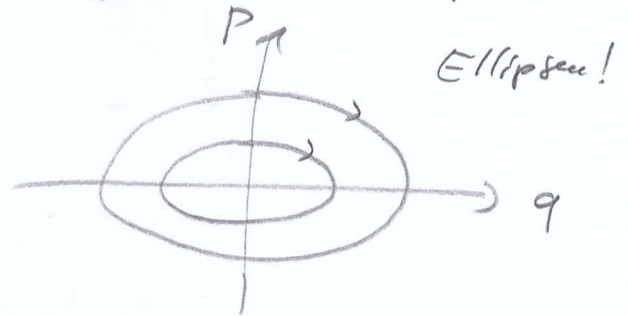
$$(v) \quad H = p \dot{q} - L = \frac{p^2}{m} - \frac{m}{2} \left(\frac{p^2}{m^2} - \omega^2 q^2 \right) = \frac{p^2}{2m} + \frac{m}{2} \omega^2 q^2$$

$$\frac{\partial L}{\partial t} = 0: \text{sklonovos System} \Rightarrow E = H = T + V = \text{const}$$

$$\Rightarrow \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\omega^2}} = 1$$

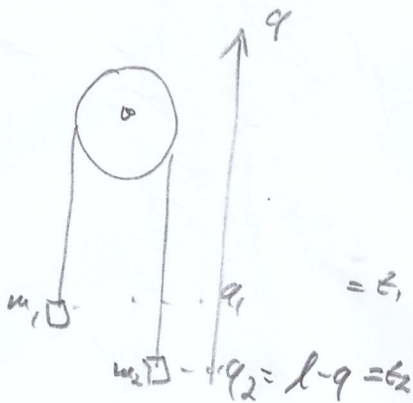
Kreisbachern:

$$a = \sqrt{\frac{2E}{m\omega^2}}, \quad b = \sqrt{2mE}$$



$$(vi) \quad \left. \begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -m\omega^2 q \\ \ddot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \end{aligned} \right\} \text{definiert Rechteck in } (q, p)\text{-Ebene}$$

(c) Atwoodsche Fallmaschine: (vgl. 3. VL, Skript d, VL 4 S 126)



$$(i) - (iii) \quad L = T - V = \frac{1}{2} (m_1 + m_2) \dot{q}^2 + m_1 g q + m_2 g (l - q)$$

$$(iv) \quad p = \frac{\partial L}{\partial \dot{q}} = (m_1 + m_2) \dot{q}$$

$$(v) \quad H = \dot{q} p - L = \frac{p^2}{m_1 + m_2} - \frac{m_1 + m_2}{2} \frac{p^2}{(m_1 + m_2)} - m_1 g q - m_2 g (l - q)$$

$$= \frac{p^2}{2(m_1 + m_2)} - m_1 g q - m_2 g (l - q)$$

$$(vi) \quad \left. \begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = m_1 g - m_2 g = (m_1 - m_2) g \\ \ddot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m_1 + m_2} \end{aligned} \right\} \text{check: } \ddot{q} = \frac{\dot{p}}{m_1 + m_2} = \frac{m_1 - m_2}{m_1 + m_2} g$$

Hamilton formalism

idea: instead of $(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$ choose $(q_1, \dots, q_f, p_1, \dots, p_f)$
with $\frac{\partial L}{\partial \dot{q}_j} = p_j$

5.1 Legendre transformation

idea: $x \rightarrow u = \frac{dx}{dt} \rightarrow$ inverse: $\varphi(u) = x$
 $y = f(x)$

$$y \Rightarrow z = xu - f(x) = x \frac{dx}{dt} - f(x) = \varphi(u)u - f(\varphi(u)) = g(u)$$

applied to Lagrangian:

$$y = L, \quad x = \dot{q} \Rightarrow u = \frac{\partial L}{\partial \dot{q}} = p \Rightarrow z = \dot{q}p - L(q, \dot{q}, t) = H(q, p, t)$$

$$\text{Similarly: } H(q_1, \dots, q_f, p_1, \dots, p_f, t) = \sum_{j=1}^f \dot{q}_j p_j - L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$$

5.2 Hamilton equations

calculate dH and equate term dq_j, dp_j, dt

$$\Rightarrow \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$$

approximation protocol

(i) $q_1, \dots, q_f$

(ii) $\underline{r}_j = \underline{r}_j(q_1, \dots, q_f, t), \quad \dot{\underline{r}}_j = \dot{\underline{r}}_j(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$

(iii) $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t) \Rightarrow$ (iv) $p_j = \frac{\partial L}{\partial \dot{q}_j}$

(v) $H = \sum_{j=1}^f \dot{q}_j p_j - L$

(vi) $\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$

$$\dot{p}_\varphi = 0 \Rightarrow \dot{p}_\varphi = m r^2 \dot{\varphi} = \text{const} \quad \text{Drehimpuls erhalten}$$

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$$\dot{p}_z = 0 \Rightarrow p_z = \text{const} = 0 \quad (\text{ebene Bewegung})$$

(b) 1D harmonischer Oszillator

$$(i) - (iii) \quad L = \frac{m}{2} (\dot{q}^2 - \omega^2 q^2)$$

$$(iv) \quad p = \frac{\partial L}{\partial \dot{q}} = m \dot{q}$$

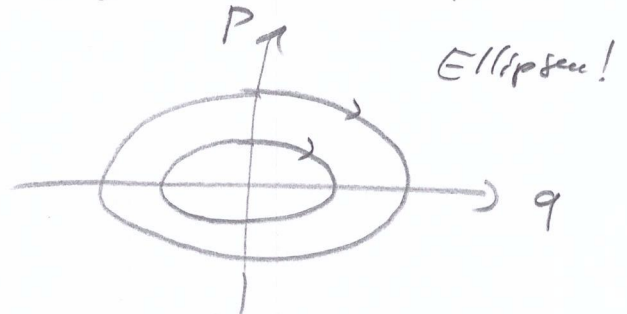
$$(v) \quad H = p \dot{q} - L = \frac{p^2}{m} - \frac{m}{2} \left(\frac{p^2}{m^2} - \omega^2 q^2 \right) = \frac{p^2}{2m} + \frac{m}{2} \omega^2 q^2$$

$$\frac{\partial L}{\partial t} = 0: \text{sklonovacee sytsem} \Rightarrow E = H = T + V = \text{const}$$

$$\Rightarrow \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\omega^2}} = 1$$

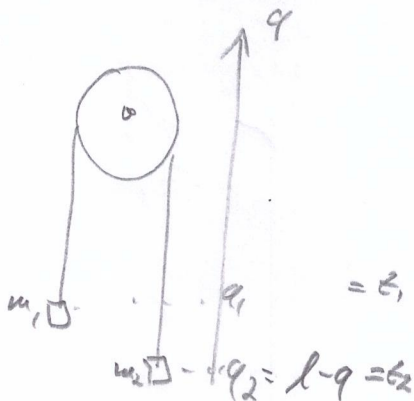
Kreisbachern:

$$a = \sqrt{\frac{2E}{m\omega^2}}, \quad b = \sqrt{2mE}$$



$$(vi) \quad \left. \begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -m\omega^2 q \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \end{aligned} \right\} \text{definiert Rechteck in } (q, p)\text{-Ebene}$$

(c) Atwoodsche Fallmaschine: (vgl. 3. K, Skript d, K4 S 126)



$$(i) - (iii) \quad L = T - V = \frac{1}{2} (m_1 + m_2) \dot{q}^2 - m_1 g q - m_2 g (l - q)$$

$$(iv) \quad p = \frac{\partial L}{\partial \dot{q}} = (m_1 + m_2) \dot{q}$$

$$(v) \quad H = \dot{q} p - L = \frac{p^2}{m_1 + m_2} - \frac{m_1 + m_2}{2} \frac{p^2}{(m_1 + m_2)^2} + m_1 g q + m_2 g (l - q)$$

$$= \frac{p^2}{2(m_1 + m_2)} + m_1 g q + m_2 g (l - q)$$

$$(vi) \quad \left. \begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -(m_1 g - m_2 g) = (m_2 - m_1) g \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m_1 + m_2} \end{aligned} \right\} \text{check: } \ddot{q} = \frac{\dot{p}}{m_1 + m_2} = -\frac{m_1 - m_2}{m_1 + m_2} g$$

1.3 Symplektische Struktur des Phasenraums

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Beobachtung: Koordinaten (q, p) nicht eindeutig
Verallgemeinert

vgl.: Forminvarianz der Lagrange-Gleichungen $\bar{L}(Q, \dot{Q}, t) = L(q(Q, \dot{Q}), t)$
(3.3 Eichinvarianz) $+ \frac{d}{dt} H(q, p, t)$

Frage: Aus welchen Transformations sind die Hamiltonschen Gleichungen
forminvariant? $(q, p) \rightarrow (Q, P)$

$$\hookrightarrow \dot{q}_j = \frac{\partial L}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial L}{\partial q_j} \rightarrow \dot{Q}_j = \frac{\partial \bar{H}}{\partial P_j}, \quad \dot{P}_j = -\frac{\partial \bar{H}}{\partial Q_j}$$

Idee: Für zyklische Variablen: Finde geeignete Transform. des (q, p) , s.d.
alle Q zyklisch sind:

Bsp: reduziertes 2-Körperproblem: $L(r, \dot{r}, \varphi, \dot{\varphi}) = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$

$$\varphi \text{ zyklisch} \Rightarrow \frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = P_\varphi = \ell = \text{const}$$

$$\text{Hamiltonsche Gleichungen: } P_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \rightarrow \dot{r} = \frac{P_r}{m}$$

$$P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \Rightarrow \dot{\varphi} = \frac{P_\varphi}{m r^2}$$

$$\Rightarrow H = P_r \dot{r} + P_\varphi \dot{\varphi} - L = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r) = \frac{P_r^2}{2m} + \frac{\ell^2}{2mr^2} + V(r)$$

Def: **Kanonische Transform.** sind diffeomorphe Transformationen
(umkehrbar, 2-to-2)

$$(q, p) \rightarrow (Q, P)$$

$$H(q, p, t) \rightarrow \bar{H}(Q, P, t)$$

und die ^{form} Hamiltonschen Gleichungen invariant lassen

Bsp: Hamiltonsches Prinzip

$$\{q, p, H\}: \int_{t_1}^{t_2} dt L = \int_{t_1}^{t_2} dt \left\{ \sum_{j=1}^L \dot{q}_j p_j - H(q, p, t) \right\} = 0$$

$$\{Q, P, \bar{H}\}: \int_{t_1}^{t_2} dt \left\{ \sum_{j=1}^L \dot{Q}_j P_j - \bar{H}(Q, P, t) \right\} = 0$$

$$\Rightarrow \sum_{j=1}^L \dot{q}_j p_j - H(q, p, t) = \sum_{j=1}^L \dot{Q}_j P_j - \bar{H}(Q, P, t) + \frac{d}{dt} H_1(q, Q, t)$$

Erzeugende des
kanon. Transform.

$$\text{Bew: } \frac{d}{dt} H_1(q, Q, t) = \sum_{j=1}^L \left\{ \frac{\partial H_1}{\partial q_j} \dot{q}_j + \frac{\partial H_1}{\partial Q_j} \dot{Q}_j \right\} + \frac{\partial H_1}{\partial t}$$

$$\Rightarrow \sum_{j=1}^L \left(p_j - \frac{\partial H_1}{\partial q_j} \right) \dot{q}_j = \sum_{j=1}^L \left(P_j + \frac{\partial H_1}{\partial Q_j} \right) \dot{Q}_j + H - \bar{H} + \frac{\partial H_1}{\partial t}$$

$$\Rightarrow p_j = \frac{\partial H_1}{\partial q_j}, \quad P_j = -\frac{\partial H_1}{\partial Q_j}, \quad \bar{H} = H + \frac{\partial H_1}{\partial t}$$

$$\begin{array}{ccc} \begin{array}{c} \rightarrow Q_j(q, p, t) \\ \text{umkehrig} \end{array} & \begin{array}{c} \text{H}_1(q, Q, t) \\ \text{Legendre-Transform.} \end{array} & \begin{array}{c} q_j(Q, P, t) \\ \downarrow \end{array} \\ P_j(q, p, t) & & P_j(Q, P, t) \end{array}$$

Weitere erzeugende Funktionen/Klammern:

$$\bullet H_2(q, P, t) = H_1(q, Q, t) - \sum_{j=1}^L \frac{\partial H_1}{\partial Q_j} Q_j \quad (\text{Legendre-Transform.})$$

$$\Rightarrow p_j = \frac{\partial H_2}{\partial q_j}, \quad Q_j = \frac{\partial H_2}{\partial P_j}, \quad \bar{H} = H + \frac{\partial H_2}{\partial t}$$

$$\bullet M_3(p, Q, t) = M_1(q, Q, t) - \sum_{j=1}^k \underbrace{\frac{\partial M_1}{\partial q_j}}_{p_j} q_j$$

$$\Rightarrow q_j = -\frac{\partial M_3}{\partial p_j} \quad p_j = -\frac{\partial M_3}{\partial Q_j}, \quad \bar{H} = H + \frac{\partial H}{\partial t}$$

$$\bullet M_4(p, P, t) = M_1(q, Q, t) - \sum_{j=1}^k \left(\underbrace{\frac{\partial M_1}{\partial q_j}}_{p_j} q_j + \underbrace{\frac{\partial M_1}{\partial Q_j}}_{-P_j} Q_j \right)$$

$$\Rightarrow q_j = -\frac{\partial M_4}{\partial p_j} \quad Q_j = -\frac{\partial M_4}{\partial P_j} \quad \bar{H} = H + \frac{\partial M_4}{\partial t}$$

Bsp ii) harmonischer Oszillator: $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$

$$M_1(q, Q) = \frac{m\omega}{2} q^2 \cot Q$$

$$\Rightarrow \left. \begin{aligned} p &= \frac{\partial M_1}{\partial q} = m\omega q \cot Q \\ P &= -\frac{\partial M_1}{\partial Q} = \frac{m\omega}{2} \frac{q^2}{\sin^2 Q} \end{aligned} \right\} \begin{aligned} q &= \left(\frac{2P}{m\omega} \right)^{1/2} \sin Q \\ p &= m\omega \left(\frac{2P}{m\omega} \right)^{1/2} \cos Q \end{aligned} \quad \frac{\cos Q}{\sin Q}$$

$$= (2m\omega P)^{1/2} \cos Q$$

$$\Rightarrow \bar{H} = H + \frac{\partial M_1}{\partial t} = \frac{1}{2m} 2m\omega P \cos^2 Q + \frac{m\omega^2}{2} \frac{2P}{m\omega} \sin^2 Q$$

$$= \omega P (\cos^2 Q + \sin^2 Q) = \omega P$$

$$\Rightarrow \bar{H}(Q, P) = \omega P \quad \text{Oszillator!} \Rightarrow \dot{P} = -\frac{\partial \bar{H}}{\partial Q} = 0 \Rightarrow P = \text{const} = \alpha$$

$$\dot{Q} = \frac{\partial \bar{H}}{\partial P} = \omega \Rightarrow Q = \omega t + \beta$$

$$\Rightarrow q = \sqrt{\frac{2\alpha}{m\omega}} \sin(\omega t + \beta)$$

iii) $M_1(q, Q, t) = -\sum_j q_j Q_j \Rightarrow p_j = \frac{\partial M_1}{\partial q_j} = -Q_j, \quad P_j = -\frac{\partial M_1}{\partial Q_j} = q_j$

iii) $M_2(q, P, t) = \sum_j q_j P_j \Rightarrow p_j = \frac{\partial M_2}{\partial q_j} = P_j, \quad Q_j = \frac{\partial M_2}{\partial P_j} = q_j$

Fazit: q und p können ineinander transformiert werden

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=> neue Ableitungen:

$$\underline{x} = \begin{pmatrix} q \\ p \end{pmatrix} : \text{Vektoren *Phasenraum*}$$

$$\underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} : \text{Ableitungsvektoren}$$

mit $\underline{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ vertauscht den Phasenraum / *vertauschen* *Teiles* folgt:

kanonische Gleichungen $\underline{\dot{x}} = \underline{J} \underline{H}_x \Leftrightarrow \begin{matrix} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{matrix}$

$$\Leftrightarrow -\underline{J} \underline{\dot{x}} = \underline{H}_x$$

Eigenschaften: $\underline{J}^2 = -\mathbb{1}$, $\underline{J}^{-1} = \underline{J}^T = -\underline{J}$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\underline{J} \quad \underline{J}^T \quad \mathbb{1}$

allgemein:

$$\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_f \\ p_1 \\ \vdots \\ p_f \end{pmatrix}, \quad \underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_f} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_f} \end{pmatrix} \quad \underline{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \left. \begin{matrix} \text{ } \\ \text{ } \end{matrix} \right\} \begin{matrix} f \\ f \end{matrix}$$

$$\Rightarrow \underline{\dot{x}} = \underline{J} \underline{H}_x \Leftrightarrow -\underline{J} \underline{\dot{x}} = \underline{H}_x$$

5.3 Symplectic structure of the phase space

Sta

idea! find ^{canonical} transformations that leave the Hamiltonian equations $\dot{q}_j = \frac{\partial H(q_1, \dots, q_n, p_1, \dots, p_n)}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

form in variant: $(q, p) \rightarrow (P, Q)$

• core Hamiltonian principle: $\int_{t_1}^{t_2} dL = \int_{t_1}^{t_2} dt \left\{ \sum_{j=1}^n p_j \dot{q}_j - H(q, p, t) \right\} = 0$
 4 equivalent forms of generating functions of canonical transformations:

(i) $M_1(q, Q, t)$: $p_j = \frac{\partial M_1}{\partial q_j}$, $P_j = -\frac{\partial M_1}{\partial Q_j}$

$\bar{H} = H + \frac{\partial M_1}{\partial t}$

$\Rightarrow \frac{\partial p_j}{\partial Q_k} = \frac{\partial^2 M_1}{\partial Q_k \partial q_j} = -\frac{\partial P_k}{\partial q_j}$

(ii) $M_2(q, P, t)$: $p_j = \frac{\partial M_2}{\partial q_j}$, $Q_j = \frac{\partial M_2}{\partial P_j}$

$= M_1(q, Q, t) - \sum_{j=1}^n \frac{\partial M_1}{\partial Q_j} Q_j$
 Legendre
 bzgl. Q_j

$\Rightarrow \frac{\partial p_j}{\partial P_k} = \frac{\partial^2 M_2}{\partial P_k \partial q_j} = \frac{\partial Q_k}{\partial q_j}$

(iii) $M_3(P, Q, t)$: $q_j = -\frac{\partial M_3}{\partial P_j}$, $P_j = -\frac{\partial M_3}{\partial Q_j}$

$= M_1 - \sum_{j=1}^n \frac{\partial M_1}{\partial q_j} q_j$
 Legendre
 bzgl. q_j

$\Rightarrow \frac{\partial q_j}{\partial Q_k} = -\frac{\partial^2 M_3}{\partial Q_k \partial P_j} = \frac{\partial P_k}{\partial P_j}$

(iv) $M_4(P, P, t)$: $q_j = -\frac{\partial M_4}{\partial P_j}$, $Q_j = \frac{\partial M_4}{\partial P_j}$

Legendre bzgl.
 Q_j and q_j

$\Rightarrow \frac{\partial q_j}{\partial P_k} = -\frac{\partial^2 M_4}{\partial P_k \partial P_j} = -\frac{\partial Q_k}{\partial P_j}$

$$\underline{x} = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \quad \underline{J} = \begin{pmatrix} 0 & \underline{1} \\ -\underline{1} & 0 \end{pmatrix}$$

$$\Rightarrow \dot{\underline{x}} = \underline{J} \underline{H}_x \Leftrightarrow \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\underline{J}^2 = -\underline{1}, \quad \underline{J}^{-1} = \underline{J}^T = -\underline{J}$$

$$\Rightarrow \underline{J} \dot{\underline{x}} = \underline{H}_x$$

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 mit $\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_k \\ p_1 \\ \vdots \\ p_l \end{pmatrix}$, $\underline{y} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_k \\ P_1 \\ \vdots \\ P_l \end{pmatrix}$ und $M_{\alpha\beta} = \frac{\partial x_\alpha}{\partial y_\beta}$, $(M^{-1})_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}$ folgt:
 $\alpha, \beta = 1, \dots, 2l$

$$M_{\alpha\beta} = \sum_{\mu, \nu=1}^{2l} \int_{\alpha\mu} \int_{\beta\nu} (M^{-1})_{\nu\mu}$$

$$\Leftrightarrow \underline{M} = \underline{J} (\underline{J} \underline{M}^{-1})^T \Leftrightarrow \underline{M}^T \underline{J} \underline{M} = \underline{J}$$

metrischer Tensor

$\underline{J}$ „symplekt. Metrik“ über das verallgemeinerte schiefssymmetrische Skalarprodukt
 $(\underline{x}, \underline{y}) = \underline{x}^T \underline{J} \underline{y} = \sum_{i,k=1}^{2l} x_i \cdot J_{ik} y_k$
 (Erzeugen der kanonischen Transform.)

(i) schiefssymmetrisch: $(\underline{y}, \underline{x}) = -(\underline{x}, \underline{y})$

(ii) bilinear: $(\underline{x}, \lambda_1 \underline{y}_1 + \lambda_2 \underline{y}_2) = \lambda_1 (\underline{x}, \underline{y}_1) + \lambda_2 (\underline{x}, \underline{y}_2)$

(iii) nicht entartet: $(\underline{x}, \underline{y}) = 0 \ \forall \underline{y} \Rightarrow \underline{x} = 0$

(iv) $(\underline{x}, \underline{x}) = 0$ Selbstorthogonal

Fazit: Die (Form-)Invarianz der kanonischen Gleichungen

$$\underline{\dot{x}} = \underline{J} \underline{H}_x$$

unter kanonischen Transformations kann

durch die symplektische Struktur o. Phaseraum

beschrieben werden.

$$\dot{y}_j = \sum_k \frac{\partial y_j}{\partial x_k} \dot{x}_k$$

$$\Rightarrow \underline{\dot{y}} = \underline{M}^{-1} \underline{\dot{x}} = \underline{J}^{-1} \underline{M}^T \underline{J} \underline{J}^{-1} \underline{M}^T \underline{J} \underline{H}_x = \underline{J} \underline{H}_y$$

$$\frac{\partial H}{\partial y_j} = \sum_k \frac{\partial H}{\partial x_k} \frac{\partial x_k}{\partial y_j}$$

$$\Leftrightarrow \underline{H}_y = \underline{M}^T \underline{H}_x$$

$\underline{H}_x = (\underline{M}^T)^{-1} \underline{H}_y$

$$\underline{M}^{-1} \underline{M} = (\underline{J}^{-1} \underline{M}^T \underline{J}) \underline{M} = \underline{J}^{-1} (\underline{M}^T \underline{J} \underline{M}) \stackrel{\underline{J}^{-1} \underline{J} = \underline{1}}{=} \underline{1}$$

$$\underline{M} = \underline{J} (\underline{J} \underline{M}^{-1})^T$$

$$\underline{M} = \begin{pmatrix} \frac{\partial q_1}{\partial q_1} & \frac{\partial q_1}{\partial q_2} & \dots & \frac{\partial q_1}{\partial q_f} & \frac{\partial q_1}{\partial p_1} & \dots & \frac{\partial q_1}{\partial p_f} \\ \frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial q_2} & \dots & \frac{\partial q_2}{\partial q_f} & \frac{\partial q_2}{\partial p_1} & \dots & \frac{\partial q_2}{\partial p_f} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_f}{\partial q_1} & \frac{\partial q_f}{\partial q_2} & \dots & \frac{\partial q_f}{\partial q_f} & \frac{\partial q_f}{\partial p_1} & \dots & \frac{\partial q_f}{\partial p_f} \\ \hline \frac{\partial p_1}{\partial q_1} & \frac{\partial p_1}{\partial q_2} & \dots & \frac{\partial p_1}{\partial q_f} & \frac{\partial p_1}{\partial p_1} & \dots & \frac{\partial p_1}{\partial p_f} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial p_f}{\partial q_1} & \frac{\partial p_f}{\partial q_2} & \dots & \frac{\partial p_f}{\partial q_f} & \frac{\partial p_f}{\partial p_1} & \dots & \frac{\partial p_f}{\partial p_f} \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \frac{\partial q}{\partial q} & \frac{\partial q}{\partial p} \\ \hline \frac{\partial p}{\partial q} & \frac{\partial p}{\partial p} \end{array} \right)$$

$$\underline{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$$\underline{J} (\underline{J} \underline{M}^{-1})^T = \begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} \left[\begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial q} & \frac{\partial q}{\partial p} \\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial p} \end{pmatrix} \right]^T$$

$$= \begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} \left[\begin{pmatrix} \underline{0} & \underline{1} \\ \underline{1} & \underline{0} \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial q} & \frac{\partial q}{\partial p} \\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial p} \end{pmatrix} \right]^T$$

$$= \begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} \begin{pmatrix} \frac{\partial p}{\partial q} & \frac{\partial p}{\partial p} \\ -\frac{\partial q}{\partial q} & -\frac{\partial q}{\partial p} \end{pmatrix}^T$$

$$= \begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} \begin{pmatrix} \left(\frac{\partial p}{\partial q}\right)^T & \left(\frac{\partial p}{\partial p}\right)^T \\ \left(\frac{\partial p}{\partial q}\right)^T & \left(\frac{\partial p}{\partial p}\right)^T \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial p}{\partial p}\right)^T & -\left(\frac{\partial q}{\partial p}\right)^T \\ -\left(\frac{\partial p}{\partial q}\right)^T & \left(\frac{\partial q}{\partial q}\right)^T \end{pmatrix}$$

Test: Oben links Quadrant

$$\frac{\partial q_i}{\partial p_k} = - \left(\frac{\partial p_j}{\partial p_k} \right)^T = \frac{\partial p_k}{\partial p_j} \quad (M_3)$$

oben rechts:

$$\frac{\partial q_i}{\partial p_k} = - \left(\frac{\partial q_j}{\partial p_k} \right)^T = - \frac{\partial q_k}{\partial p_j} \quad (M_4)$$

unten links:

$$\frac{\partial p_j}{\partial q_k} = - \left(\frac{\partial p_j}{\partial q_k} \right)^T = - \frac{\partial p_k}{\partial q_j} \quad (M_1)$$

unten rechts:

$$\frac{\partial p_j}{\partial p_k} = \left(\frac{\partial q_j}{\partial q_k} \right)^T = \frac{\partial q_k}{\partial q_j} \quad (M_2)$$

$$\underline{M} = \underline{J} (\underline{J} \underline{M}')^T \quad | \cdot \underline{J} \quad (\underline{J}^2 = \underline{1})$$

$$\underline{J} \underline{M} = -\underline{1} (\underline{J} \underline{M}')^T$$

$$= - (\underline{M}')^T \underline{J}^T \quad | \cdot \underline{M}^T$$

$$\underline{M}^T \underline{J} \underline{M} = - \underbrace{\underline{M}^T (\underline{M}')^T}_{(\underline{M}^{-1} \underline{M})^T = \underline{1}} \underline{J}^T = -\underline{J}^T = \underline{J}$$

$$\Rightarrow \underline{M}^T \underline{J} \underline{M} = \underline{J}$$

Def: Die Menge der Matrizen $\underline{M}$ (kann auch $\mathbb{J}$ sein)
 mit $\underline{M}^T \underline{J} \underline{M} = \underline{J}$ bildet die reelle **symplektische**
Gruppe $\mathbb{J}$ über $\mathbb{R}^{2k}$ (Symplektische Gruppe der symplektischen
 Struktur)

§. 3 symplectic structure of the phase space (continued) §5a
 again the idea: transformations that leave $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

form invariant: $(q, p) \rightarrow (Q, P)$

$$H(q, p, t) \rightarrow \bar{H}(Q, P, t)$$

variable transform:

$$\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_f \\ p_1 \\ \vdots \\ p_f \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_f \\ P_1 \\ \vdots \\ P_f \end{pmatrix}$$

$$\dot{\underline{x}} = \underline{J} \underline{H}_x$$

$$\Rightarrow M_{xp} = \frac{\partial x}{\partial y_p}, \quad M_{yp} = \frac{\partial y}{\partial x_p}$$

$$\Rightarrow \underline{M} = \underline{J} (\underline{J} \underline{M}^{-1})^T \iff \underline{M}^T \underline{J} \underline{M} = \underline{J}, \quad \underline{J} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

non-degenerate, bi-linear

• $\underline{J}$ defines a skew-symmetric form : $(\underline{u}, \underline{v}) = -(\underline{v}, \underline{u})$
 alternating form $= \underline{u}^T \underline{J} \underline{v}$

$$\bullet (\underline{u}, \underline{u}) = 0 \Rightarrow 0 = (\underline{u} + \underline{u}, \underline{u} + \underline{u}) = \underbrace{(\underline{u}, \underline{u})}_{=0} + \underbrace{(\underline{u}, \underline{u})}_{=0} + \underbrace{(\underline{u}, \underline{u})}_{=0} + \underbrace{(\underline{u}, \underline{u})}_{=0}$$

$$\bullet \dot{y}_j = \sum_{k=1}^{2f} \frac{\partial y_j}{\partial x_k} \dot{x}_k \quad \Rightarrow \underline{\dot{y}} = \underline{M}^{-1} \underline{\dot{x}} = \dots = \underline{J} \underline{H}_y \quad (\dot{\underline{x}} = \underline{J} \underline{H}_x)$$

$$\frac{\partial H}{\partial y_j} = \sum_{k=1}^{2f} \frac{\partial H}{\partial x_k} \frac{\partial x_k}{\partial y_j} \quad \left. \vphantom{\frac{\partial H}{\partial y_j}} \right\} \underline{H}_y = \underline{M}^T \underline{H}_x$$

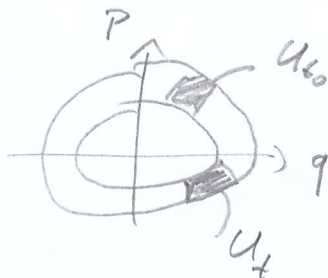
$$\underline{M} = \left(\begin{array}{c|c} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \hline \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{array} \right)_{2f} \iff \begin{pmatrix} M_3(p, Q, t) & M_4(p, P, t) \\ \hline M_1(q, Q, t) & M_2(q, P, t) \end{pmatrix}$$

Def: The set of matrices $\underline{M}$ (canonical transformations) with $\underline{M}^T \underline{J} \underline{M} = \underline{J}$
 forms the (real) symplectic group $\mathbb{S}$ over $\mathbb{R}^{2f}$.

Satz: Bei der Hamiltonschen Zeitentwicklung ist das

Phasenraumvolumen erhalten.

Phasenraumvolumen:



Hamiltonsche Zeitentwicklung: $\dot{\underline{x}} = \underline{f} = \underline{f}(\underline{x})$

$\hookrightarrow \underline{x}(t, t_0, \underline{x}_0) =: \phi_{t, t_0}(\underline{x}_0)$ Fluss im Phasenraum
Aufg. bed.

$\phi_{t, t_0}: \Gamma \rightarrow \Gamma$ } beschreibt die Zeitentwicklung der Anfangswerte
 $\underline{x}_0(t_0) \mapsto \underline{x}(t)$ } Anfangswerte

Bsp.: 1D harmonischer Oszillator:

$(q, p) \in \Gamma$, $\begin{cases} \dot{q} = p \\ \dot{p} = -\omega_0^2 q \end{cases} \Rightarrow \dot{\underline{x}} = \underline{A} \underline{x} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$

$\Rightarrow \underline{x}(t) = \phi_{t, t_0}(\underline{x}_0) = \exp\left(\frac{(t-t_0)}{1} \underline{A}\right) \underline{x}_0$

$= \left(\sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \underline{A}^n \right) \underline{x}_0$

$= \left(\cos(\omega_0(t-t_0)) + \frac{\underline{A}}{\omega_0} \sin(\omega_0(t-t_0)) \right) \underline{x}_0$

$= \begin{pmatrix} \cos(\omega_0(t-t_0)) & \frac{1}{\omega_0} \sin(\omega_0(t-t_0)) \\ -\omega_0 \sin(\omega_0(t-t_0)) & \cos(\omega_0(t-t_0)) \end{pmatrix} \underline{x}_0$

$\underline{P}(t-t_0)$

$\begin{cases} \underline{A}^2 = -\omega_0^2 \underline{1} \\ \underline{A}^{2n} = (-1)^n \omega_0^{2n} \underline{1} \\ \underline{A}^{2n+1} = (-1)^n \omega_0^{2n+1} \frac{1}{\omega_0} \underline{A} \end{cases}$



Beweis: Menge von Lösungen (x_0) ist

Phasenraumgebiet U_{t_0} mit Volumen $V_{t_0} = \int d^{2k} x_0$

$$\Rightarrow V_t = \int_{U_t} d^{2k} x = \int_{U_{t_0}} d^{2k} x_0 \det \left(\frac{\partial x}{\partial x_0} \right) = \int_{U_{t_0}} d^{2k} x_0 \det \left(D\phi_{t,t_0}(x_0) \right)$$

Functional determinant
Jacobian Matrix

$$(D\phi_{t,t_0})_{ik} = \frac{\partial \phi_{t,t_0}^i(x_0)}{\partial x_0^k} = \frac{\partial x^i}{\partial x_0^k}$$

$t \approx t_0$: $\phi_{t,t_0}(x_0) = x_0 + \underbrace{\bar{F}(x_0, t)}_{\substack{\text{H} \\ = -\frac{\partial H}{\partial t}}}(t-t_0) + O((t-t_0)^2)$

$$\Rightarrow \frac{\partial \phi_{t,t_0}^i(x_0)}{\partial x_0^k} = \delta_{ik} + \frac{\partial \bar{F}^i}{\partial x_0^k} (t-t_0) + O((t-t_0)^2)$$

mit $\det(\underline{1} + \underline{B}\epsilon) \approx 1 + \epsilon \operatorname{Sp}(\underline{B}) + O(\epsilon^2)$:

$$\det(D\phi_{t,t_0}) = 1 + (t-t_0) \sum_{i=1}^{2k} \frac{\partial \bar{F}^i}{\partial x_0^i}$$

$$\underbrace{\operatorname{div} \bar{F}}_{= \frac{\partial}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial}{\partial p} \frac{\partial H}{\partial q}} = 0$$

$$\Rightarrow V_t = \int_{U_{t_0}} d^{2k} x_0 (1 + O(t-t_0)^2) \approx V_{t_0}$$

Bsp.: 1D harmonische Osk.

$$x'(t) = \sum_{k=1}^2 P_{ik} x_0^k \quad \text{mit } \det \underline{P} = \cos^2(\omega_0(t-t_0)) + \sin^2(\omega_0(t-t_0)) = 1$$

$$\Rightarrow \underbrace{dx^1 dx^2}_{\substack{\text{Fl} \\ \text{Element}}} = (\det \underline{P}) dx_0^1 dx_0^2 = dx_0^1 dx_0^2$$

$$\hookrightarrow \int_{U_t} d^{2k} x = \int_{U_{t_0}} d^{2k} x_0$$

Jede **Observable** (phys. Größe, Energie, Impuls ...) lässt sich als Funktion von q, p, t schreiben: $g(q, p, t)$

$\Rightarrow$ Zeitliche Änderung des Wertes $g(t), p_i(t)$ im Phasenraum:

$$\begin{aligned} \frac{dg}{dt} &= \sum_{i=1}^f \left(\frac{\partial g}{\partial q_i} \dot{q}_i + \frac{\partial g}{\partial p_i} \dot{p}_i \right) + \frac{\partial g}{\partial t} \\ &= \sum_{i=1}^f \left(\frac{\partial g}{\partial p_i} \frac{\partial H}{\partial q_i} - \frac{\partial g}{\partial q_i} \frac{\partial H}{\partial p_i} \right) + \frac{\partial g}{\partial t} \\ &=: \{g, H\} + \frac{\partial g}{\partial t} \end{aligned}$$

Def: Für zwei beliebige Observable $g(q, p, t), h(q, p, t)$ heißt

$$\{g, h\} = \sum_{i=1}^f \left(\frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right) \quad \text{Poisson-Klammer}$$

$$\begin{aligned} \bullet \quad \{g, h\} &= \begin{pmatrix} g_{,1} & g_{,2} \end{pmatrix} = \begin{pmatrix} g_{,1} & g_{,2} \end{pmatrix}^T \begin{pmatrix} h_{,1} \\ h_{,2} \end{pmatrix} = \sum_{i=1}^f \frac{\partial g}{\partial x_i} \sum_{k=1}^f \frac{\partial h}{\partial x_k} \\ &= \begin{pmatrix} \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial q} \\ \frac{\partial h}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial q} & \frac{\partial g}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial p} \\ -\frac{\partial h}{\partial q} \end{pmatrix} \end{aligned}$$

$$\bullet \quad \{g, h\} = -\{h, g\} \quad \text{Schiefsymmetrie}$$

$$\{g, h_1 + h_2\} = \{g, h_1\} + \{g, h_2\}$$

$$\{g, h\} = 0 \quad \forall h \Rightarrow g = \text{const.}$$

$$\{g, g\} = 0$$

Symplektisch,

Skalarprodukt!

Satz: Die Poisson-Klammer sind invariant unter

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Kanonical Transformations:

$$\text{Beweis: } \frac{\partial f}{\partial x_i} = \sum_k \frac{\partial f}{\partial y_k} \frac{\partial y_k}{\partial x_i} = \sum_k M_{ki}^{-1} \frac{\partial f}{\partial y_k} \Rightarrow \underline{f}_x = (\underline{M}^{-1})^T \underline{f}_y$$

$$\Rightarrow \underline{f}_x^T = \underline{f}_y^T (\underline{M}^{-1})$$

$$\{f, h\} = (\underline{f}_{y, h_y}) \underline{f}_x^T \underline{f} \underline{h}_x = \underline{f}_y^T \underline{M}^{-1} \underline{f} (\underline{M}^{-1})^T \underline{h}_y$$

$$= \underline{f}_y^T \underline{f} \underline{h}_y = (\underline{f}_{y, h_y}) = \{f, h\}$$

Satz: Die Transformation $(q, p) \rightarrow (Q, P)$ ist genau dann kanonisch,

wenn $\{Q_i, P_j\} = \delta_{ij}$

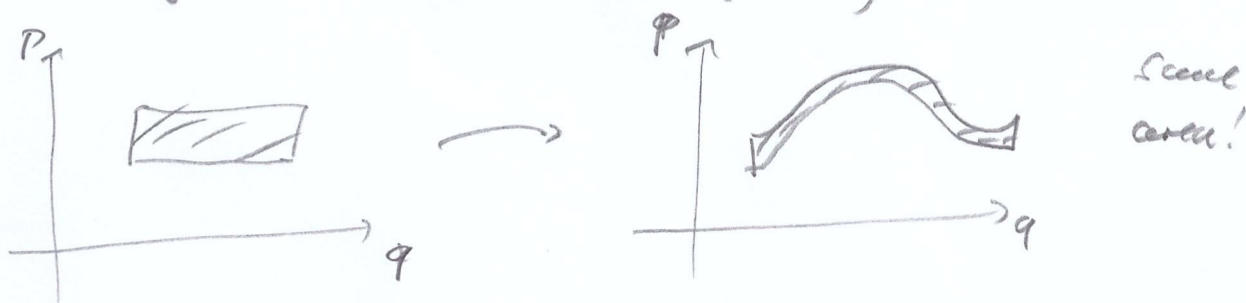
$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0$$

5.4 Liouville's Theorem

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Thm: The flow of Hamiltonian systems preserves the phase space volume.

(The phase-space distribution function is constant along the trajectories of Hamiltonian systems.)



• flow: $\phi_{t,t_0} : \mathcal{M} \rightarrow \mathcal{M}$
 $x_0(t_0) \mapsto \phi_{t,t_0}(x_0) = x(t)$

5.5 Poisson brackets

given 2 functions / observables $g(q, p, t)$ and $h(q, p, t)$, their

Poisson bracket takes the form:

$$\{g, h\} = \sum_{i=1}^f \left(\frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right)$$

• $\{g, h\} = -\{h, g\}$

• $\{g, \lambda_1 h_1 + \lambda_2 h_2\} = \lambda_1 \{g, h_1\} + \lambda_2 \{g, h_2\}$

• $\{g, h\} = 0 \quad \forall h \Rightarrow g = \text{const.}$

• $\{g, g\} = 0$

• $\frac{dg}{dt} = \{g, H\} + \frac{\partial g}{\partial t}$

• $\{g, h\} = (\underline{g}_x, \underline{h}_x) = \underline{g}_x^T \underline{J} \underline{h}_x$, $\underline{g}_x = \begin{pmatrix} \frac{\partial g}{\partial q} \\ \frac{\partial g}{\partial p} \end{pmatrix}$, $\underline{J} = \begin{pmatrix} 0 & \underline{1} \\ -\underline{1} & 0 \end{pmatrix}$, $\underline{h}_x = \begin{pmatrix} \frac{\partial h}{\partial q} \\ \frac{\partial h}{\partial p} \end{pmatrix}$

Eigenschaften:

$g = g(q, p)$: nicht explizit zeit abhängig ($\frac{\partial g}{\partial t} = 0$)

$$\Rightarrow \frac{dg}{dt} = \{g, H\}$$

• g : Bewegungskonstante: $(\frac{dg}{dt} = 0) \Leftrightarrow \{g, H\} = 0$

• $g = q_k$ oder $g = p_k$

$$\dot{q}_k = \{q_k, H\} = \sum_{i=1}^f \left(\underbrace{\frac{\partial q_k}{\partial q_i}}_{\delta_{ik}} \frac{\partial H}{\partial p_i} - \underbrace{\frac{\partial q_k}{\partial p_i}}_{=0} \frac{\partial H}{\partial q_i} \right) = \frac{\partial H}{\partial p_k}$$

$$\dot{p}_k = \{p_k, H\} = \sum_{i=1}^f \left(\underbrace{\frac{\partial p_k}{\partial q_i}}_{=0} \frac{\partial H}{\partial p_i} - \underbrace{\frac{\partial p_k}{\partial p_i}}_{\delta_{ik}} \frac{\partial H}{\partial q_i} \right) = - \frac{\partial H}{\partial q_k}$$

$$\text{kompakt: } \dot{x}_k = \{x_k, H\} = \sum_{i=1}^{2f} \underbrace{\frac{\partial x_k}{\partial x_i}}_{\delta_{ki}} \underbrace{f_{ij}}_{\frac{\partial H}{\partial x_j}} = \sum_{j=1}^{2f} f_{kj} \frac{\partial H}{\partial x_j}$$

$$\text{oder } \dot{x} = \underline{F} H_x$$

• Fundamentale Poisson-Klammern

$$\left. \begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \right\} \{x_i, x_j\} = \sum_{k,l=1}^{2f} \underbrace{\frac{\partial x_i}{\partial x_k}}_{\delta_{ik}} \underbrace{f_{kl}}_{\frac{\partial H}{\partial x_l}} = f_{ij}$$

• Satz: Die Transformationskanon $(q, p) \Leftrightarrow (Q, P)$ ist genau dann

kanonisch, wenn $\{Q_i, P_j\} = \delta_{ij}, \{Q_i, Q_j\} = 0 = \{P_i, P_j\}$.

Beweis: $\dot{x} = \underline{F} H_x$

$$\begin{aligned} M^T \underline{F} M &= \underline{F} \\ M &= \underline{F} (\underline{F}^T M)^T \end{aligned}$$

$$\left. \begin{aligned} \dot{y}_k &= \sum_{l=1}^f f_{kl} \frac{\partial H}{\partial y_l} \\ M_{kl} &= \frac{\partial q_k}{\partial y_l}, \dot{y} = M^T \dot{x} \end{aligned} \right\} \Leftrightarrow \{y_k, y_l\} = f_{kl} \quad \text{Beweis alt H4}$$

Folgende Aussagen sind äquivalent:

1. Die Transformation $x \mapsto y$ ist kanonisch.
2. Die Gleichungen $\dot{x} = JH_x$ sind invariant.
3. Die Poisson-Klammern $\{g, h\}$ sind invariant für alle g, h .
4. Die fundamentalen Poisson-Klammern $\{x_i, x_j\} = \delta_{ij}$ sind invariant.
5. Die Jacobi-Matrix $M_{\alpha\beta} = \frac{\partial x_\alpha}{\partial y_\beta}$ ist symplektisch, d.h.: $M^T J M = J$.
6. Es existiert eine Erzeugende.

Bezug zur Quantenmechanik:

Klass. Observable: $g(q, p, t) \rightarrow$ ^{Kohärent Raum} $\hat{g}: \mathcal{H} \rightarrow \mathcal{H}$

Poisson-Klammer $\{f, g\} \rightarrow \frac{1}{i\hbar} [\hat{f}, \hat{g}] = \frac{1}{i\hbar} (\hat{f}\hat{g} - \hat{g}\hat{f})$
Kommutator

Fundamentale Poisson-Klammern:

$$\{q_i, p_j\} = \delta_{ij} \rightarrow [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \rightarrow [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

Klass. Hamilton-Funktion $H(q, p, t) \rightarrow$ Hamilton-Operatoren: $\hat{H}(\hat{q}, \hat{p})$

Bewegungsgleichungen:

Heisenberg'sche Bewegungsgleichungen

$$\frac{dg}{dt} = \{g, H\} + \frac{\partial g}{\partial t} \rightarrow \frac{d\hat{g}}{dt} = \frac{1}{i\hbar} [\hat{g}, \hat{H}] + \frac{\partial \hat{g}}{\partial t}$$

5.6 Kanonical-Jacobi-Theorie

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Idee: Finde Trasse, die alle Koordinaten zyklisch macht.
Kanonsche

etwa (erforderliche Fall): $\bar{H} \equiv 0$

Wähle: $H_2(q, p, t) =: S(q, p, t) \quad (q, p) \rightarrow (Q, P)$

$$H(q, p, t) \rightarrow \bar{H}(Q, P, t) = H + \frac{\partial S}{\partial t}$$

$$\text{wähle } P_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial p_k}$$

$$\text{so dass } \bar{H} = H(q_1, \dots, q_f, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_f}, t) + \frac{\partial S}{\partial t} = 0$$

Kanonical-Jacobi-Gleichung

$$\rightarrow \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k} = 0 \quad \Rightarrow Q_k = \text{const} = \beta_k$$

$$\dot{P}_k = -\frac{\partial \bar{H}}{\partial Q_k} = 0 \quad \Rightarrow P_k = \text{const} = \alpha_k \Rightarrow S(q, \underline{\alpha}, t)$$

Parameter!

Kurzversion:

$$(i) \quad H(q, p, t), \quad P_k = \frac{\partial S}{\partial q_k} \Rightarrow H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$$

$$(ii) \quad \text{Löse Kanonical-Jacobi-Gleichung} \Rightarrow S(q, \underline{\alpha}, t), \quad \underline{\alpha} = \underline{P}$$

$$(iii) \quad \text{Aus Erzeugenden } S(q, \underline{\alpha}, t) \text{ folgt: } Q_k = \frac{\partial S(q, \underline{\alpha}, t)}{\partial \alpha_k} = \beta_k$$

$$\Rightarrow q_j = q_j(\underline{\alpha}, \underline{\beta}, t)$$

$$(iv) \quad P_j = \frac{\partial S}{\partial q_j} = P_j(q, \underline{\alpha}, t) = P_j(q(\underline{\alpha}, \underline{\beta}), \underline{\alpha}, t)$$

(v) Bestimmung von $\underline{\alpha}, \underline{\beta}$ aus Anfangsbedingungen

$$\left. \begin{array}{l} q_j(0) = q_j(\underline{\alpha}, \underline{\beta}, 0) \\ p_j(0) = p_j(\underline{\alpha}, \underline{\beta}, 0) \end{array} \right\} \Rightarrow \underline{\alpha} = \underline{\alpha}(q(0), p(0))$$

$$\underline{\beta} = \underline{\beta}(q(0), p(0))$$

Berechnung von S:

$$\frac{dS(q, \dot{q}, t)}{dt} \stackrel{\substack{\uparrow \\ \text{const}}}{=} \sum_{k=1}^K \frac{\partial S}{\partial \dot{q}_k} \dot{q}_k + \frac{\partial S}{\partial t} \stackrel{\substack{\uparrow \\ H=K=K}}{=} \sum_{k=1}^K p_k \dot{q}_k - H = L$$

$\Rightarrow S = \int L dt$ Wirkung formal entlang Trajektorie
(wobei unbekannt)

Bsp: 1D harmonischer Oszillator:

(i) $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$

$S(q, P, t)$ mit $P = \frac{\partial S}{\partial q}$

$\Rightarrow \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + \frac{m}{2} \omega^2 q^2 + \frac{\partial S}{\partial t} = 0$

(ii) Ansatz: $S(q, P, t) = W(q; P) + V(t; P)$ Separation ansatz nach q, t.
Parameter

$\Rightarrow \frac{1}{2m} \left(\frac{dW}{dq} \right)^2 + \frac{m}{2} \omega^2 q^2 = - \frac{dV}{dt} = \alpha = \text{const}$
unabhängig nach t

$\Rightarrow V(t) = -\alpha t + V_0$

(b) $\left(\frac{dW}{dq} \right)^2 = m^2 \omega^2 \left(\frac{2\alpha}{m\omega^2} - q^2 \right) \Rightarrow W = m\omega \int dq \sqrt{\frac{2\alpha}{m\omega^2} - q^2}$
 $S = W + V$

$\Rightarrow S(q, \dot{q}, t) = m\omega \int dq \sqrt{\frac{2\alpha}{m\omega^2} - q^2} - \alpha t + V_0 = \alpha t + m\omega \left[\frac{q}{2} \left(\frac{2\alpha}{m\omega^2} - q^2 \right) \right]$

(iii) $Q = \frac{\partial S}{\partial \dot{q}} \stackrel{\substack{\uparrow \\ k=P}}{=} -t + \frac{1}{\omega} \int dq \frac{1}{\sqrt{\frac{2\alpha}{m\omega^2} - q^2}} \stackrel{\substack{\uparrow \\ \text{O.P.d.A.}}}{=} + \frac{1}{m\omega} \arcsin \left(q \sqrt{\frac{m\omega^2}{2\alpha}} \right)$
 $= -t + \frac{1}{\omega} \arcsin \left(q \omega \sqrt{\frac{m}{2\alpha}} \right) \stackrel{!}{=} \beta = \text{const}$
 $[Q] = \text{Zeit}, \beta = t_0$

$$\Rightarrow q = \frac{1}{\omega} \sqrt{\frac{2\alpha}{m}} \sin(\omega(t+\beta))$$

$$\begin{aligned} \text{(iv)} \quad p &= \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{dW}{dq} = m \omega \sqrt{\frac{2\alpha}{m\omega^2} - q^2} \\ &= \sqrt{2\alpha m} \cos(\omega(t+\beta)) \end{aligned}$$

$$\text{(v)} \quad t=t_0=0, p(0)=0, q(0)=q_0 \neq 0$$

$$\begin{aligned} \Rightarrow q_0 &= \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega t) \\ 0 = p_0 &= \sqrt{2\alpha m} \cos(\omega t) \Rightarrow p = \frac{\pi}{2\omega} \end{aligned} \quad \left. \vphantom{\begin{aligned} \Rightarrow q_0 &= \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega t) \\ 0 = p_0 &= \sqrt{2\alpha m} \cos(\omega t) \end{aligned}} \right\} \Rightarrow q_0 = \sqrt{\frac{2\alpha}{m\omega^2}}$$

$$\Rightarrow \alpha = \frac{m}{2} \omega^2 q_0^2 = E \quad (\text{max. amplitude} \Rightarrow \text{max. pot. Energie})$$

$$\left. \begin{aligned} \Rightarrow L = P = E & \quad (\text{Energie}) \\ Q = t_0 & \quad (\text{Zeit}) \end{aligned} \right\} \text{keine verallgem. Koordinaten}$$

Spezial: $\frac{\partial H}{\partial \alpha} = 0 \Leftrightarrow \frac{\partial H}{\partial \alpha} \Big|_{H, H_0} = 0$, H : Integrationsbereich.

$$H(q, \frac{\partial \mathcal{L}}{\partial \dot{q}}, t) + \frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \text{Lagrange: } \mathcal{H}(q, p, t) = W(q, p) - Et$$

$$\Rightarrow H(q, \frac{\partial W}{\partial q}) = E$$

$$\begin{aligned} \bullet \quad p_j &= \frac{\partial W}{\partial q_j}, \quad q_j = \frac{\partial W}{\partial p_j}, \quad \bar{H} = H = E \Rightarrow \dot{q}_j = \frac{\partial \bar{H}}{\partial p_j} = \frac{\partial \mathcal{L}}{\partial p_j} = \omega_j \\ &\quad q_j = \omega_j t + p_j = \frac{\partial W}{\partial p_j} \end{aligned}$$

kanon. Erzeugende

reproduziert die kanonische Trajektorien

5.5 Poisson brackets (continued)

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- $g = g(q, p, t) \Rightarrow \frac{dg}{dt} = \{g, H\}$

- fundamental Poisson brackets:

$$\left. \begin{aligned} \{q_i, q_j\} &= 0 = \{p_i, p_j\} \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \right\} \Leftrightarrow \{x_i, x_j\} = J_{ij}$$

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

- Theorem: The transformation $(q, p) \rightarrow (\underline{Q}, \underline{P})$ is canonical iff $\{Q_i, P_j\} = \delta_{ij}$, $\{Q_i, Q_j\} = 0 = \{P_i, P_j\}$.

→ Proof?

- Unsupp: equivalent statements:

(i) $X \mapsto Y$ is canonical

(ii) $\underline{\dot{x}} = \underline{J} \underline{H_x}$ are invariant.

(iii) $\{q, h\}$ are invariant.

(iv) $\{x_i, x_j\} = \delta_{ij}$ are invariant

(v) Jacobian $M_{dp} = \frac{\partial \underline{x}}{\partial \underline{p}}$ is symplectic ($M^T J M = J$)

(vi) There exists a generating function

- link to QM

§.6 Hamilton-Jacobi theory

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idea: find canonical transform for which all variables become canonical.

$$\hookrightarrow \bar{H} \equiv 0$$

$$\hookrightarrow H_2(q, p, t) =: S(q, p, t) \text{ with: } (q, p) \rightarrow (Q, P)$$

$$H(q, p, t) \rightarrow \bar{H}(Q, P, t) = H + \frac{\partial S}{\partial t}$$

$$p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial p_k}$$

$$\Rightarrow \bar{H} = H(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}, t) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k} = 0$$

$$\Rightarrow Q_k = P_k = \text{const.}$$

"parameters"

$$\dot{P}_k = -\frac{\partial \bar{H}}{\partial Q_k} = 0$$

$$\Rightarrow P_k = \alpha_k = \text{const.} \Rightarrow S(q, \alpha, t)$$

• recipe:

$$(i) H(q, p, t), \quad p_k = \frac{\partial S}{\partial q_k} \Rightarrow H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$$

$$(ii) \text{ Solve Hamilton-Jacobi equation } \Rightarrow S(q, \alpha, t); \quad \alpha = \underline{P}$$

$$(iii) Q_k = \frac{\partial S}{\partial \alpha_k} = p_k \Rightarrow q_j = q_j(\alpha, t)$$

$$(iv) P_j = \frac{\partial S}{\partial q_j} = p_j(q, \alpha, t)$$

(v) Determine α, β from initial conditions:

$$\left. \begin{aligned} q_j(0) &= q_j(\alpha, \beta, 0) \\ p_j(0) &= p_j(q(0), \alpha, \beta, 0) \end{aligned} \right\} \Rightarrow \alpha = \alpha(q(0), p(0))$$

$$\beta = \beta(q(0), p(0))$$

• $S = \int L dt$ 'action'

$$\frac{\partial H}{\partial t} = 0 \Leftrightarrow \frac{dH}{dt} = \{H, H\} = 0 \Rightarrow H: \text{Integral of motion} \Rightarrow S(q, p, t) = W(q) - Et$$

$$H(q(0), p(0)) = E$$

$$S(q,t) = W(q) - Et \quad (\text{Wirkungswellen})$$

↳ Phase φ festgeschrieben: $\varphi \sim W$

↳ Teilchenimpuls: $p = \nabla W$

$$\Rightarrow H(q, p) = \frac{1}{2m} (\nabla W)^2 + V(q) = E$$

↓
Schrödinger-Gleichung: $\underbrace{\left(-\frac{\hbar^2}{2m} \Delta + V(x) \right)}_{\hat{H}} \psi(x) = E \psi(x)$

$$q \rightarrow x$$

$$p \rightarrow \frac{\hbar}{i} \nabla$$

↑
 $\psi(x) = e^{\frac{i}{\hbar} W(x)}$ Wellenfunktion

Idee: ausgedehnte, starre Körper (vgl. deformierbare Medien)
Elastomechanik / Hydrodynamik

↳ Def: Starrkörper:

(a) System von N Masspunkten mit festen Abständen
(↳ Zwangsbedingungen) $M = \sum_{i=1}^N m_i$

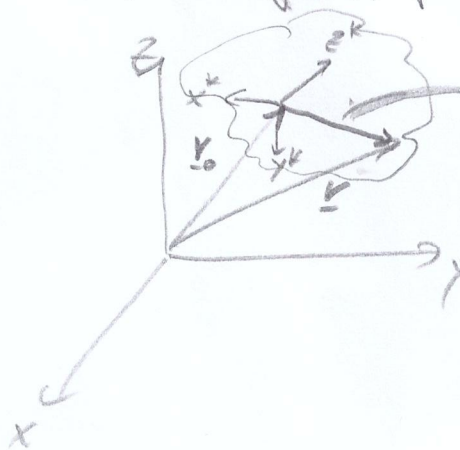
(b) vorgegeben, kontinuierliche Massenverfölg: $\rho(\mathbf{r})$,
(↳ Gesamtmasse: $M = \int \rho(\mathbf{r}) d^3r$)

Beschreibung: (i) **Raumfests Koordinatensystem** $S: (x, y, z), (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z), (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$
↳ Inertialsystem

(ii) **Körperfestes Koordinatensystem** $S^*: (x^*, y^*, z^*), (\mathbf{e}_x^*, \mathbf{e}_y^*, \mathbf{e}_z^*), (\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*)$
↳ feststehend / kein Inertialsystem
↳ Ursprung: **Schwerpunkt** (kann gedreht sein)
(kann) (auch beliebiger Bezugspunkt)

(iii) **Schwerpunktsystem** $S': (x', y', z'), (\mathbf{e}_x', \mathbf{e}_y', \mathbf{e}_z'), (\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$
↳ translationale von S (gleiche Richtung)

Freiheitsgrade: $f = 6$ (vgl. 2.1)



$\mathbf{r} - \mathbf{r}_0 = \mathbf{d}$: Konstante (relativ zum Aufpunkt)
 $d = |\mathbf{r}^*| = |\mathbf{r}| \cos \alpha$ (Stamm aus * fortbewegt!)

hinzu: $\mathbf{r}_0$: Schwerpunkts Koordinate $\mathbf{r}_s$

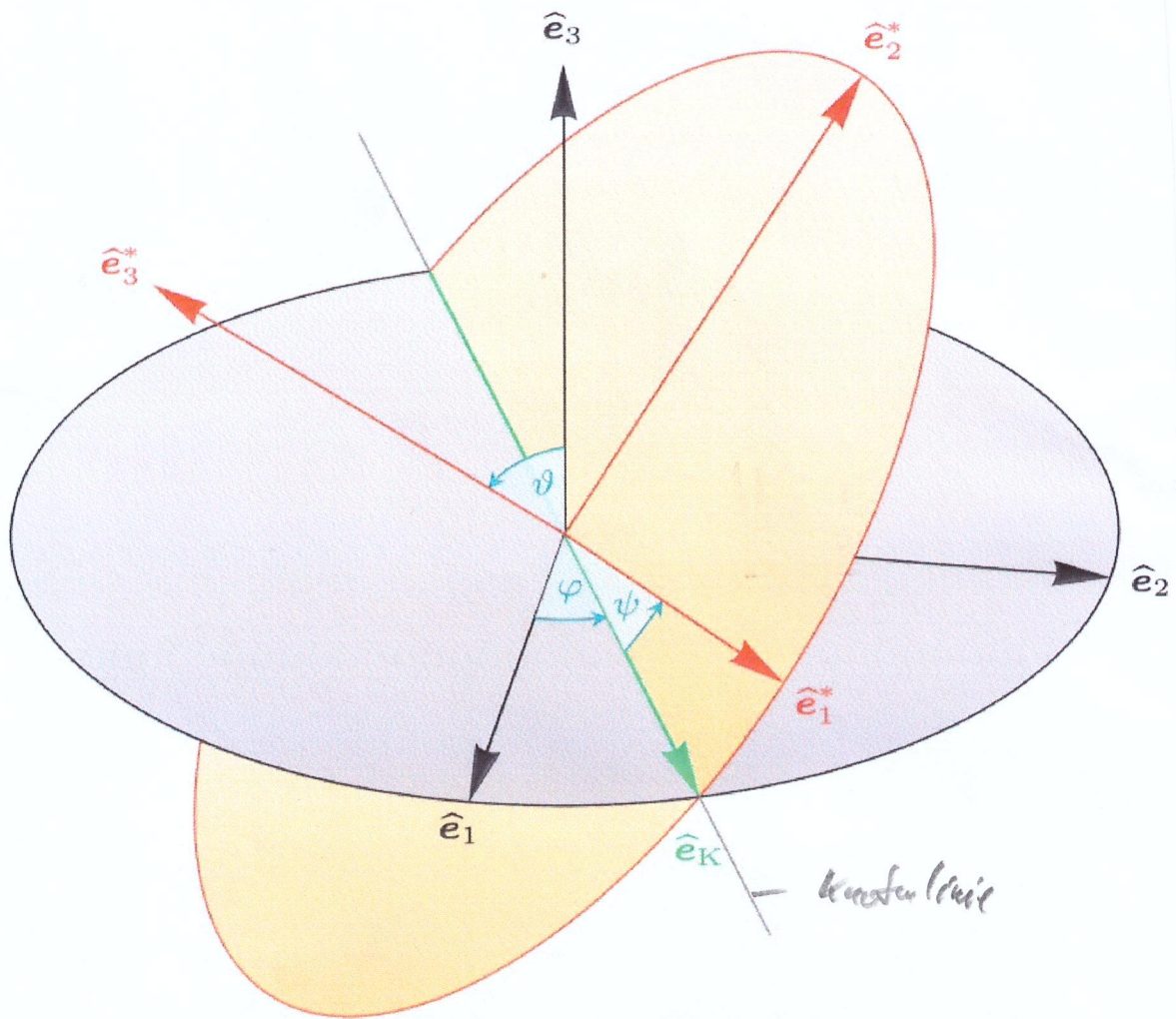
$\Rightarrow \mathbf{r} = \mathbf{r}_0 + \mathbf{d}$ $|\mathbf{d}| = \cos \alpha$ nur Drehung!

$\Rightarrow \dot{\mathbf{r}} = \dot{\mathbf{r}}_0 + \dot{\mathbf{d}} = \dot{\mathbf{r}}_0 + \underbrace{\underbrace{\boldsymbol{\omega} \times \mathbf{d}}_{\text{Rotation um Drehachse}}}_{\text{gegebener } \boldsymbol{\omega}}$

Schwerpunktsw. Aufpunkt Rotation um Drehachse
gegebenen $\boldsymbol{\omega}$

Beziehung: $S \rightarrow S^*$: Translation $S \rightarrow S'$, Rotation $S' \rightarrow S^*$ 67

Euler'scher Winkel: zur Beschreibung einer beliebigen Drehung $S' \rightarrow S^*$



φ : Winkel zwischen $\underline{e}_1$ und $\underline{e}_K$

ψ : Winkel zwischen $\underline{e}_3$ und $\underline{e}_3^*$

θ : — — — $\underline{e}_1^*$ und $\underline{e}_K$

$\Rightarrow$ (i) Drehung um φ (Z-Achse / $\underline{e}_3$)

$$R_3(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\uparrow$
Z-Achse

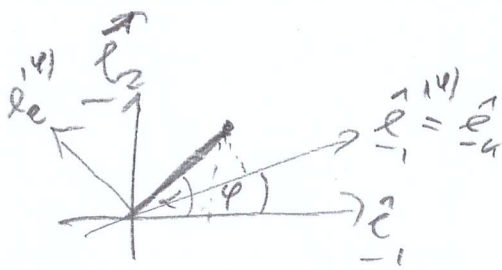
$\Rightarrow$ Basis system: $\underline{e}_1^{(1)}, \underline{e}_2^{(1)}, \underline{e}_3^{(1)} = \underline{e}_3$
 $\quad \quad \quad = \underline{e}_K$

(ii) Drehung um ψ (Knotenlinie, $\underline{e}_1^{(1)}, \underline{e}_2^{(1)}$)

$$R_1(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{pmatrix}$$

$\uparrow$
X-Achse
im 1. System

$\Rightarrow$ Basis system: $\underline{e}_1^{(2)}, \underline{e}_2^{(2)}, \underline{e}_3^{(2)}$
 $\quad \quad \quad \parallel \quad \quad \quad \parallel$
 $\quad \quad \quad \underline{e}_1^{(1)} = \underline{e}_K \quad \quad \quad \underline{e}_3^*$

Drehung um φ (Z-Rotation)

$$\begin{aligned} x_1 &= r \cos \alpha \\ x_2 &= r \sin \alpha \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1' &= r \cos(\alpha - \varphi) = r (\cos \alpha \cos \varphi + \sin \alpha \sin \varphi) \\ x_2' &= r \sin(\alpha - \varphi) = r (\sin \alpha \cos \varphi - \cos \alpha \sin \varphi) \end{aligned}$$

$$\Rightarrow x_1' = x_1 \cos \varphi + x_2 \sin \varphi$$

$$x_2' = -x_1 \sin \varphi + x_2 \cos \varphi$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

6.1 Kinetische Energie & Trägheitstensor

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(a) System von N Massenpunkten: wähle: $r_0 = r_1$

$$T = \frac{1}{2} \sum_{i=1}^N m_i \dot{\underline{r}}_i^2 = \frac{1}{2} \sum_{i=1}^N m_i \left[\frac{d}{dt} \left(\underline{r}_0 + \underline{d}^{(i)} \right) \right]^2$$

$$= \frac{1}{2} \sum_{i=1}^N m_i \left[\dot{\underline{r}}_0 + \underline{\omega} \times \underline{d}^{(i)} \right]^2$$

$$= \underbrace{\frac{1}{2} \sum_{i=1}^N m_i \dot{\underline{r}}_0^2}_M \quad + \underbrace{\frac{2}{2} \dot{\underline{r}}_0 \cdot \sum_{i=1}^N m_i (\underline{\omega} \times \underline{d}^{(i)})}_{(\underline{r}_0 \times \underline{\omega}) \cdot \sum_{i=1}^N m_i \underline{d}^{(i)}} \quad + \underbrace{\frac{1}{2} \sum_{i=1}^N m_i (\underline{\omega} \times \underline{d}^{(i)})^2}_{\text{Rotationsenergie}}$$

Schwerpunktsenergie = 0 weil Schwerpunkt im Ursprung von Σ^*

Nebenbedg: $(\underline{\omega} \times \underline{d})^2 = \omega^2 d^2 \sin^2 \alpha = \omega^2 d^2 (1 - \cos^2 \alpha)$
(skip (i))

$$= \omega^2 d^2 - \omega^2 d^2 \cos^2 \alpha$$

$$= \omega^2 d^2 - (\underline{\omega} \cdot \underline{d})^2 = (\omega_1^2 + \omega_2^2 + \omega_3^2) (d_1^2 + d_2^2 + d_3^2) - (\omega_1 d_1 + \omega_2 d_2 + \omega_3 d_3)^2$$

$$= \sum_{j,k=1}^3 \omega_j (d_j^2 \delta_{jk} - d_j d_k) \omega_k$$

$$\Rightarrow T = \frac{1}{2} M \dot{\underline{r}}_0^2 + \frac{1}{2} \sum_{j,k=1}^3 \omega_j \underbrace{\sum_{i=1}^N m_i (d_j^2 \delta_{jk} - d_j d_k)}_{\Theta_{jk}} \omega_k$$

$$= \frac{1}{2} M \dot{\underline{r}}_0^2 + \frac{1}{2} \sum_{j,k=1}^3 \omega_j \Theta_{jk} \omega_k$$

Trägheitstensor

$$= \frac{1}{2} M \dot{\underline{r}}_0^2 + \frac{1}{2} \underline{\omega}^T \underline{\Theta} \underline{\omega}$$

$$= T_{\text{Translation}} + T_{\text{Rotation}}$$

↑
im Σ' System

(b) analog Massendichte (Kontinuierlich):

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$$T = \frac{1}{2} \int d^3r \rho(\underline{r}) (\dot{\underline{r}} + \underline{\omega} \times \underline{r})^2$$

$$= \frac{1}{2} \underbrace{\int d^3r \rho(\underline{r})}_{M} \dot{\underline{r}}^2 + (\underline{r}_S \times \underline{\omega}) \cdot \underbrace{\int d^3r \rho(\underline{r}) \underline{r}}_{=0} + \frac{1}{2} \sum_{j,k} \omega_j \Theta_{jk} \omega_k$$

mit $\Theta_{jk} = \int d^3r \rho(\underline{r}) [\underline{r}^2 \delta_{jk} - r_j r_k]$

$$\Rightarrow T = \frac{1}{2} M \dot{\underline{r}}^2 + \frac{1}{2} \underline{\omega}^T \underline{\Theta} \underline{\omega}$$

6.2 Eigenschaften des Trägheitstensors

$$\Theta_{jk} = \sum_{i=1}^N m_i (d_i^{(i)2} \delta_{jk} - d_j d_k)$$

$$\Rightarrow \underline{\Theta} = \sum_{i=1}^N m_i \begin{pmatrix} d_2^{(i)2} + d_3^{(i)2} & -d_1^{(i)} d_2^{(i)} & -d_1^{(i)} d_3^{(i)} \\ -d_2^{(i)} d_1^{(i)} & d_1^{(i)2} + d_3^{(i)2} & -d_2^{(i)} d_3^{(i)} \\ d_3^{(i)} d_1^{(i)} & -d_3^{(i)} d_2^{(i)} & d_1^{(i)2} + d_2^{(i)2} \end{pmatrix}$$

analog: $\underline{r} = (x_1, x_2, x_3)$

$$\underline{\Theta} = \int d^3r \rho(\underline{r}) \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_2 x_1 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_3 x_1 & -x_3 x_2 & x_1^2 + x_2^2 \end{pmatrix}$$

6. Mechanics of a rigid body

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idea: spatially extended system of

(a) N masses with fixed distances $M = \sum_{i=1}^N m_i$

(b) continuous density of mass $M = \int d^3r \rho(r)$

coordinates:

(i) lab system (system of inertia) : S

(ii) body-fixed coordinates system : S^*

(iii) translational system of (i) with center of mass as origin : S'

position in S^* : $\underline{r}^* = \underline{r} - \underline{r}_0$
 $\uparrow \quad \uparrow$
 position center of mass

$\Rightarrow |\underline{d}| = \text{const}$, but orientation can change

Euler angles: $\varphi, \vartheta, \varphi$

$$\underline{R}_3(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

φ : rotation around $\underline{e}_3$

ϑ : rotation around $\underline{e}_1^{(R)}$

$$\underline{R}_1(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta \\ 0 & -\sin \vartheta & \cos \vartheta \end{pmatrix}$$

φ : rotation around $\underline{e}_2^{(R)}$

$$\underline{R}_2(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ combined / chained rotation: $\underline{R}(\varphi, \vartheta, \varphi) = \underline{R}_2(\varphi) \underline{R}_1(\vartheta) \underline{R}_3(\varphi)$

6.1 kinetic energy & inertia tensor

$$(a) T = \frac{1}{2} M \dot{\underline{r}}_0^2 + \frac{1}{2} \underline{\omega}^T \underline{\Theta} \underline{\omega}$$

center of mass

rotation

$$\underline{\Theta}_{jk} = \sum_{i=1}^N m_i \left[d^{(i)2} \delta_{jk} - d_j d_k \right]$$

$$(b) T = \frac{1}{2} M \dot{\underline{r}}_0^2 + \frac{1}{2} \underline{\omega}^T \underline{\Theta} \underline{\omega}$$

$$\underline{\Theta}_{jk} = \int d^3r \rho(r) \left[d^2 \delta_{jk} - d_j d_k \right]$$

$$\begin{aligned} \Theta \text{ of Sphere } \rho = \text{const} \\ \rho(r) = \begin{cases} M/4\pi R^3 & r \leq R \\ 0 & r > R \end{cases} \\ \Theta = \Theta_1 = \Theta_2 = \Theta_3 = \frac{2}{5} MR^2 \\ \underline{\Theta} = \begin{pmatrix} \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \Theta \end{pmatrix} \end{aligned}$$

$\underline{\Theta}$ ist ein Tensor 2. Stufe

↳ Transformation unter Drehungen: $\text{Drehmatrix in } \mathbb{R}^3 \left(\underline{R}^{-1} \underline{R} = \underline{R}^T \underline{R} = \underline{1} \right)$
 $\det \underline{R} = 1$

$$\text{Koordinaten: } x_\mu \rightarrow x'_\mu = \sum_{\nu=1}^3 R_{\mu\nu} x_\nu$$

$$\Rightarrow \underline{\Theta}_{\mu\nu} \rightarrow \underline{\Theta}'_{\mu\nu} = \sum_{\lambda=1}^3 \sum_{\sigma=1}^3 R_{\mu\lambda} R_{\nu\sigma} \underline{\Theta}_{\lambda\sigma}$$

Kompakt: $\underline{x} \rightarrow \underline{x}' = \underline{R} \underline{x}$

$$\underline{\Theta} \rightarrow \underline{\Theta}' = \underline{R}^T \underline{\Theta} \underline{R} \quad (\text{Äquivalenztrafo})$$

Bsp.: 1. Stufe (Vektor) $A'_\mu = \sum_{\lambda=1}^3 R_{\mu\lambda} A_\lambda$

2. Stufe (Matrix) $A'_{\mu\nu} = \sum_{\lambda,\sigma=1}^3 R_{\mu\lambda} R_{\nu\sigma} A_{\lambda\sigma}$

n. Stufe $A'_{\mu\nu\dots\zeta} = \sum_{\lambda,\sigma,\dots,\tau=1}^3 R_{\mu\lambda} R_{\nu\sigma} \dots R_{\zeta\tau} A_{\lambda\sigma\dots\tau}$

Trafo nachweis: $\underline{\Theta}_{\mu\nu} = \sum_{i=1}^n m_i \left[\left(\sum_{\alpha=1}^3 x_\alpha^{(i)2} \right) \delta_{\mu\nu} - x_\mu^{(i)} x_\nu^{(i)} \right] = \dots = \underline{\Theta}'_{\mu\nu}$

mit $x'_\mu = \sum_{\lambda=1}^3 R_{\mu\lambda} x_\lambda \Rightarrow \sum_{\alpha=1}^3 x_\alpha'^2 = \sum_{\alpha=1}^3 \sum_{\lambda=1}^3 \sum_{\sigma=1}^3 R_{\alpha\lambda} R_{\alpha\sigma} x_\lambda x_\sigma$

$$= \sum_{\lambda=1}^3 \sum_{\sigma=1}^3 \underbrace{\sum_{\alpha=1}^3 R_{\alpha\lambda} R_{\alpha\sigma}}_{\delta_{\lambda\sigma}} x_\lambda x_\sigma$$

$$= \sum_{\lambda=1}^3 x_\lambda^2 \quad (\text{invariant})$$

iii) $\sum_{\lambda,\sigma=1}^3 R_{\mu\lambda} R_{\nu\sigma} \delta_{\lambda\sigma} = \sum_{\lambda=1}^3 R_{\mu\lambda} R_{\nu\lambda}^T = \delta_{\mu\nu} \quad (\text{invariant})$

$$\Rightarrow \sum_{\lambda,\sigma=1}^3 R_{\mu\lambda} R_{\nu\sigma} \underline{\Theta}_{\lambda\sigma} = \sum_{i=1}^n m_i \sum_{\lambda=1}^3 \sum_{\sigma=1}^3 R_{\mu\lambda} R_{\nu\sigma} \left(x_\lambda^{(i)2} \delta_{\lambda\sigma} - x_\lambda^{(i)} x_\sigma^{(i)} \right)$$

$$= \sum_{i=1}^n m_i \left(x_\mu^{(i)2} \delta_{\mu\nu} - x_\mu^{(i)} x_\nu^{(i)} \right) = \underline{\Theta}'_{\mu\nu} \quad \text{Tensor ist invariant.}$$

$\underline{\Theta}$ ist diagonalisierbar durch eine orthogonale

Transformation $\underline{R} \in SO(3)$ (spezielle orthogonale Gruppe in $\mathbb{R}^3$,
 $\hookrightarrow$ alle Drehungen

$$\underline{\Theta}' = \overset{\underline{R}^{-1}}{\underline{R}^T} \underline{\Theta} \underline{R} = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix} \quad \text{d.h. gestrecktes Koordinatensystem!} \\ (y_1, y_2, y_3)$$

in Richtung der Hauptträgheitsachsen

$$\underline{\Theta}' = \int d^3y \, S(\underline{y}) \begin{pmatrix} y_2^2 + y_3^2 & 0 & 0 \\ 0 & y_1^2 + y_3^2 & 0 \\ 0 & 0 & y_1^2 + y_2^2 \end{pmatrix}$$

$$\Rightarrow \Theta_i \geq 0, \quad i=1,2,3.$$

Diagonalisierung $\Rightarrow$ Eigenwertproblem: $\underline{\Theta} \overset{\uparrow}{\underline{\hat{\omega}}^{(i)}} = \Theta_i \underline{\hat{\omega}}^{(i)}$
Hauptachsenträgheitsmomente
 (denn ist Diagonal)

$$\Leftrightarrow \det(\underline{\Theta} - \Theta_i \underline{1}) = 0$$

$\Rightarrow$ 3 reelle, positive (semi-) definite Eigenwerte Θ_i .

$\underline{\Theta}$ ist linear in $S(r)$: additiv beim Zusammensetzen 2er Körper

Θ_i Hauptträgheitsmomente

$\Theta_1 \neq \Theta_2 \neq \Theta_3$ ungleichachsiger Körper

$\Theta_1 = \Theta_2 \neq \Theta_3$ Symmetrischer Körper (Kreiszylinder)

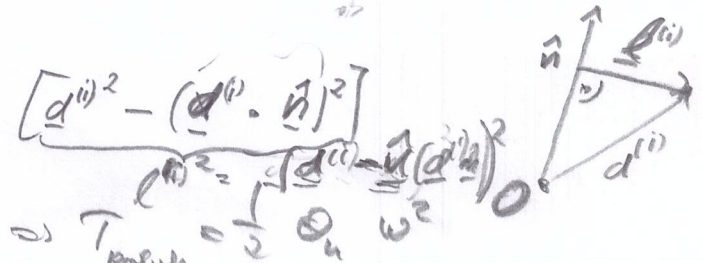
$\Theta_1 = \Theta_2 = \Theta_3$ "Kugelsymmetrie" (nicht notwendig Kugelkörper!)

Trägheitsmoment

Bzgl. Drehachse $\underline{\omega} = \omega \underline{\hat{n}}$

$$\Theta_n = \sum_{i,j=1}^3 \hat{n}_i \Theta_{ij} \hat{n}_j = \underline{\hat{n}}^T \underline{\Theta} \underline{\hat{n}} = \sum_{i=1}^N m_i \left[d^{(i)2} - (\underline{d}^{(i)} \cdot \underline{\hat{n}})^2 \right]$$

$$= \sum_{i=1}^N m_i \ell^{(i)2} \Rightarrow T_{\text{rot}} = \frac{1}{2} \Theta_n \omega^2$$



Sei $O_{\mu v}$ der Trägheits tensor im ebenen Schwerpunkt S

Zentrierter Körperfesten System S , Sei S' ein zu S achsen-
paralleles, um a verschobenes System. Dann gilt:

$$\Theta' = \Theta_{\mu v} + M(a^2 \delta_{\mu v} - a_{\mu} a_{\nu})$$

(Trägheits tensor bzgl. $\vec{u}$: $\Theta_u' = \Theta_u + M l^2$, $l^2 = a^2 - (\vec{a} \cdot \vec{u})^2$, $\Theta_u = \vec{u} \cdot \vec{\Theta} \cdot \vec{u}$)

Beweis: Transformation $\underline{\Theta} \rightarrow \underline{\Theta}'$ mit $\underline{r}' = \underline{r} + \underline{a}$

Bsp.: Kugelsymmetrische Kernverteilung $\rho(r) = \rho(r)$

$$\Rightarrow \Theta_1 = \Theta_2 = \Theta_3 =: \Theta$$

$$3\Theta = \sum_{i=1}^3 \Theta_i = \int dV \rho(r) \left[(x_2^2 + x_3^2) + (x_3^2 + x_1^2) + (x_1^2 + x_2^2) \right]$$

$$= 2 \cdot 4\pi \int_0^R dr r^4 \rho(r) \cdot 2r^2$$

$$\Rightarrow \Theta = \frac{8\pi}{3} \int_0^R dr r^4 \rho(r)$$

Für homogenen Kern verteilt: $M = \underbrace{4\pi}_V R^3 \rho(r)$

$$\Theta = \frac{2}{R^3} M \int_0^R dr r^4 = \frac{2}{5} MR^2$$

(ii) Abrollende Kugel:

Θ bzgl. unserem Drehachsendurch A :

$$\Theta_A = \Theta + MR^2$$

$$= \frac{2}{5} MR^2 + MR^2$$

$$= \frac{7}{5} MR^2$$



6.3 Drehimpuls & Bewegungsgleichungen

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$$\begin{aligned}
 (a) \quad \underline{L} &= \sum_{i=1}^N m_i \underline{r}_i \times \dot{\underline{r}}_i = \sum_{i=1}^N m_i (\underline{r}_0 + \underline{d}^{(i)}) \times (\dot{\underline{r}}_0 + \underline{\omega} \times \underline{d}^{(i)}) \\
 &= \underbrace{\sum_{i=1}^N m_i (\underline{r}_0 \times \dot{\underline{r}}_0)}_M + \underbrace{\sum_{i=1}^N m_i \underline{d}^{(i)} \times \dot{\underline{r}}_0}_{=0} + \underline{r}_0 \times (\underline{\omega} \times \underbrace{\sum_{i=1}^N m_i \underline{d}^{(i)}}_{=0}) \\
 &\quad + \underbrace{\sum_{i=1}^N m_i \underline{d}^{(i)} \times (\underline{\omega} \times \underline{d}^{(i)})}_{\text{relativ Drehimpuls } \underline{L}_R \text{ (Eigendrehimpuls)}}
 \end{aligned}$$

Schwerpunkt (Bewegungspunkt)

$$\begin{aligned}
 (b) \quad \underline{L} &= \underbrace{\underline{r}_0 \times M \dot{\underline{r}}_0}_{\underline{L}_S} + \int d^3r \, \rho(\underline{r}) \underline{r} \times (\underline{\omega} \times \underline{r}) \\
 &= \underline{L}_R = \int d^3r \, \rho(\underline{r}) [\underline{r}^2 \underline{\omega} - (\underline{r} \cdot \underline{\omega}) \underline{r}]
 \end{aligned}$$

"Bewegungspunkt" des Schwerpunkts

b(a,c,d,b)

$$\Rightarrow \underline{L}_R = \underline{\Theta} \underline{\omega}$$

Komponentenweise: $L_{R\mu} = \sum_{\nu=1}^3 \int d^3r \, \rho(\underline{r}) [\underline{r}^2 \delta_{\mu\nu} - r_\mu r_\nu] \omega_\nu$

$\Theta_{\mu\nu}$

$\Rightarrow$ im Allgemeinen $\underline{L}_R \nparallel \underline{\omega}$, kann von $\underline{\omega}$ getrennt werden!

$\underline{L}_R \parallel \underline{\omega}$, wenn $\underline{\omega}$ in Richtung einer Hauptträgheitsachse liegt.

$$\underline{\Theta} = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{21} & \Theta_{22} & \Theta_{23} \\ \Theta_{31} & \Theta_{32} & \Theta_{33} \end{pmatrix}$$

Bewegungsgleichungen

(1) Schwerpunkt (11), (12), (13)

(2) Drehimpuls (21), (22), (23)

(3) Energie (31), (32), (33)

(4) Drehmoment (41), (42), (43)

(5) Drehimpuls (51), (52), (53)

(11)

(12) Drehmoment (12), (13)

(13) Drehmoment (13), (14)

(21) Drehmoment (21), (22)

(22) Drehmoment (22), (23)

(23) Drehmoment (23), (24)

(31) Drehmoment (31), (32)

(32) Drehmoment (32), (33)

6.2 Properties of inertia tensor (continued)

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$$(a) \quad \Theta_{jk} = \sum_{i=1}^N m_i \left[d^{(i)2} \delta_{jk} - d_j^{(i)} d_k^{(i)} \right]$$

$$(b) \quad \Theta_{jk} = \int d^3r \, \rho(\mathbf{r}) \left(d^2 \delta_{jk} - d_j d_k \right)$$

• diagonalization of $\underline{\Theta}$: $\underline{x} \rightarrow \underline{x}' = \underline{R} \underline{x}$

$$\underline{\Theta} \rightarrow \underline{\Theta}' = \underline{R}^T \underline{\Theta} \underline{R}$$

seek that $\underline{\Theta}' = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix}$ Θ_i : principal moments of inertia
 $\underline{\omega}^{(i)}$: principal axes

$\Rightarrow$ eigenvalue equation: $\underline{\Theta} \underline{\omega}^{(i)} = \Theta_i \underline{\omega}^{(i)}$

$\Theta_1 = \Theta_2 = \Theta_3$ spherical top (symmetric, not necessarily a sphere!)

$\Theta_1 = \Theta_2 \neq \Theta_3$ symmetric top

$\Theta_1 \neq \Theta_2 \neq \Theta_3$ asymmetric top

• moment of inertia w.r.t. $\underline{\omega} = \underline{\hat{n}} \omega$: $\Theta_n = \underline{\hat{n}}^T \underline{\Theta} \underline{\hat{n}}$

$$= \sum_{i=1}^N m_i \left[d^{(i)2} - (\underline{d}^{(i)} \cdot \underline{\hat{n}})^2 \right]$$

$$= \sum_{i=1}^N m_i \left(\ell^{(i)} \right)^2$$

$$\ell^{(i)} = \left| \underline{d}^{(i)} - \underline{\hat{n}} (\underline{d}^{(i)} \cdot \underline{\hat{n}}) \right|$$

$\Rightarrow T_{\text{rot}} = \frac{1}{2} \Theta_n \omega^2$

• Steiner's theorem: $\Theta'_{jk} = \Theta_{jk} + M (a^2 \delta_{jk} - a_j a_k)$

with rotation axes of S and S' shifted by $\underline{a}$

• $\Theta_n' = \Theta_n + M \ell^2$, $\ell^2 = a^2 - (\underline{a} \cdot \underline{\hat{n}})^2$, $\Theta_i = \underline{\hat{n}}^T \underline{\Theta} \underline{\hat{n}}$

6.3 Angular momentum & equations of motion

Feb

$$(a) \underline{L} = \underbrace{M \underline{r}_0 \times \dot{\underline{r}}_0}_{\text{center of mass}} + \underbrace{\sum_{i=1}^N m_i \underline{d}^{(i)} \times (\underline{\omega} \times \underline{d}^{(i)})}_{\text{relative angular momentum}}$$

$\underline{L}_s$: center of mass $\underline{L}_R$: relative angular momentum

$$(b) \underline{L} = M \underline{r}_0 \times \dot{\underline{r}}_0 + \int d^3v \, \rho(\underline{r}) \, \underline{d} \times (\underline{\omega} \times \underline{d})$$

$$\Rightarrow \underline{L}_R = \underline{\Theta} \underline{\omega}$$

in principle: $\underline{L}_R \neq \underline{\Theta} \underline{\omega}$ (rotation by $\underline{\Theta}$)

Wähle raumfestes System im Schwerpunkt: $\underline{r}_0 = 0$

$$\Rightarrow \underline{L} = \sum_{i=1}^N m_i \underline{d}^{(i)} \times (\underline{\omega} \times \underline{d}^{(i)}) = \underline{L}_R$$

$$\left(\text{Schwerpunkt: } \underline{r}_0 = \frac{1}{M} \sum_{i=1}^N m_i \underline{d}^{(i)} \right) \Rightarrow M \underline{r}_0 = \sum_{i=1}^N m_i \underline{d}^{(i)}$$

Newton'sche Gleichung für i -tes Teilchen: (vgl. 2.1)

$$m_i \underline{\ddot{d}}^{(i)} = \underbrace{\underline{F}^{(i)}_{\text{außen}}}_{\text{äußere Kräfte z.B. } m_i g} + \underbrace{\sum_{j=1}^N \underline{F}^{(ij)}}_{\text{innere Kräfte}}$$

$$\Rightarrow \underbrace{\sum_{i=1}^N m_i \underline{\ddot{d}}^{(i)}}_{M \underline{\ddot{r}}_0} = \sum_{i=1}^N \underline{F}^{(i)}_{\text{außen}} + \sum_{i,j=1}^N \underline{F}^{(ij)}$$

mit actio = reactio ($\underline{F}^{(ij)} = -\underline{F}^{(ji)} \Rightarrow \underline{F}^{(ij)} + \underline{F}^{(ji)} = 0$) folgt:

Schwerpunktssatz: $M \underline{\ddot{r}}_0 = \sum_{i=1}^N \underline{F}^{(i)}_{\text{außen}} =: \underline{F}^{(\text{außen})}$

Der Schwerpunkt bewegt sich so, als ob die gesamte Masse als starrer Körper in ihm versetzt wäre und die **Resultierende** $\underline{F}^{(\text{außen})}$ aller äußeren Kräfte in ihm angreift.

• innere Kräfte beeinflussen nicht den Schwerpunktsbewegung!

$$\underline{L} = \sum_{i=1}^N m_i \underline{d}^{(i)} \times \underline{\ddot{d}}^{(i)} = \sum_{i=1}^N \underline{d}^{(i)} \times \underline{F}^{(i)}_{\text{außen}} + \underbrace{\sum_{i,j=1}^N \underline{d}^{(i)} \times \underline{F}^{(ij)}}_{=0}$$

$$\frac{d}{dt} \underline{L}$$

$$\underline{F}^{(ij)} = -\underline{F}^{(ji)}$$

$$= 0$$

$$\underline{d}^{(i)} \times \underline{F}^{(ij)} + \underline{d}^{(j)} \times \underline{F}^{(ji)} = (\underline{d}^{(i)} - \underline{d}^{(j)}) \times \underline{F}^{(ij)}$$

Entlang Verbindungsvektor
zwischen i und $j \rightarrow \underline{d}^{(i)} - \underline{d}^{(j)}$

$$\Rightarrow \text{Drehimpulsatz: } \sum_{i=1}^N m_i \underline{\dot{d}}^{(i)} \times \underline{\ddot{d}}^{(i)} = \sum_{i=1}^N \underline{\dot{d}}^{(i)} \times \underline{\dot{d}}^{(i)} \text{ verschwindet}$$

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$$\Leftrightarrow \frac{d}{dt} \underline{L} = \underline{M} \quad \text{Drehmoment}$$

$$(i) \quad \frac{d}{dt} \underline{L} = \frac{d}{dt} \left(\sum_{i=1}^N m_i \underline{\dot{d}}^{(i)} \times (\underline{\omega} \times \underline{\dot{d}}^{(i)}) \right) = \frac{d}{dt} (\underline{L}_i \underline{e}_i) = \frac{d}{dt} \left(\sum_{i=1}^N \underline{\Theta}_{ij} \omega_j \underline{e}_i \right)$$

$$= \sum_{j=1}^N \underline{\dot{\Theta}}_{ij} \omega_j \underline{e}_i + \sum_{j=1}^N \underline{\Theta}_{ij} \dot{\omega}_j \underline{e}_i$$

$$= \underbrace{\sum_{j=1}^N \underline{\dot{\Theta}}_{ij} \omega_j \underline{e}_i}_{\text{Abh. f. Beobachter}} + \sum_{j=1}^N \underline{\Theta}_{ij} \omega_j \underbrace{(\underline{\omega} \times \underline{e}_i)}_{\substack{\text{Körperfeste} \\ \text{bleibt}}} = \frac{d}{dt} \underline{L} + \underline{\omega} \times \underline{L} = \underline{M}$$

Abh. f. Beobachter
des Körpersystems, $\underline{e}_i$ konstant

Körperfeste
bleibt

(ii) alternativer Treuesatz von Leibniz auf Körpersystemen ($\underline{L}_0 = 0$):

$$\frac{d}{dt} = \left(\frac{d}{dt} \right)^* + \underline{\omega} \times$$

$$\Rightarrow \frac{d}{dt} \underline{L} = \left(\frac{d}{dt} \right)^* \underline{L} + \underline{\omega} \times \underline{L} = \underline{M}$$

$$= \underline{\underline{\Theta}} \underline{\dot{\omega}} + \underline{\omega} \times \underline{\underline{\Theta}} \underline{\omega} = \underline{M}$$

$\uparrow$
fest im Körpersystem

$\Rightarrow$ Euler'sche Gleichungen: Körperfeste Achsen = Hauptträgheitsachsen: $\underline{L}_i = \underline{\Theta}_{ii} \omega_i$

$$\underline{\Theta}_1 \dot{\omega}_1 + (\underline{\Theta}_3 - \underline{\Theta}_2) \omega_2 \omega_3 = M_1$$

$$\underline{\Theta}_2 \dot{\omega}_2 + (\underline{\Theta}_1 - \underline{\Theta}_3) \omega_1 \omega_3 = M_2$$

$$\underline{\Theta}_3 \dot{\omega}_3 + (\underline{\Theta}_2 - \underline{\Theta}_1) \omega_2 \omega_1 = M_3$$

$$\Rightarrow \underline{\omega} \times \underline{\Theta} \underline{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} \underline{\Theta}_1 \omega_1 \\ \underline{\Theta}_2 \omega_2 \\ \underline{\Theta}_3 \omega_3 \end{pmatrix}$$

• nichtlinear, gekoppelt!

6.4. Beispiele
 Bsp. 1) Symmetrischer Kegel: $\Theta_1 = \Theta_2 = \Theta \neq \Theta_3$ (kugelförmig) 77
 im Körpersystem S (Struktur in folgendem weggelassen)

$$\Rightarrow \Theta_3 \dot{\omega}_3 \neq (\Theta - \Theta_3) \omega_2 \omega_3 = 0 \Rightarrow \omega_3 = \text{const.}$$

$$\Theta_1 \dot{\omega}_1 = (\Theta - \Theta_3) \omega_2 \omega_3 \Rightarrow \dot{\omega}_1 = \frac{\Theta - \Theta_3}{\Theta} \omega_2 \omega_3$$

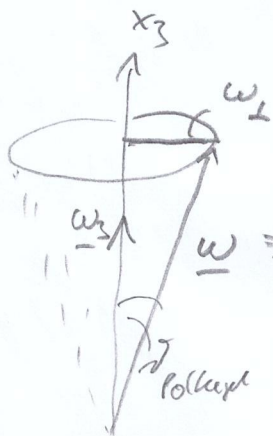
$$\Rightarrow \ddot{\omega}_1 = \frac{\Theta - \Theta_3}{\Theta} \omega_3 \dot{\omega}_2 = - \left(\frac{\Theta - \Theta_3}{\Theta} \omega_3^2 \right) \omega_1$$

$$\dot{\omega}_2 = - \frac{\Theta - \Theta_3}{\Theta} \omega_1 \omega_3$$

$$\Rightarrow \omega_1(t) = \omega_{\perp} \sin(\omega_0 t - \varphi_0) \text{ mit } \omega_0 = \frac{\Theta - \Theta_3}{\Theta} \omega_3$$

$$\omega_2(t) = -\omega_{\perp} \cos(\omega_0 t - \varphi_0), \quad \omega_3, \varphi_0 \text{ Integrationskonstanten}$$

$$\Rightarrow \omega_1^2 + \omega_2^2 = \omega_{\perp}^2 = \text{const.}$$



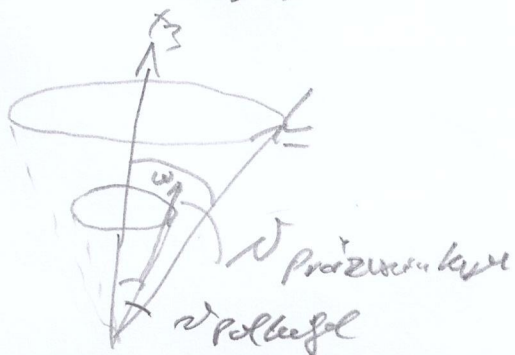
$$\Rightarrow \omega_1^2 + \omega_2^2 + \omega_3^2 = \omega_{\perp}^2 + \omega_3^2 \Rightarrow \gamma_{\text{Polkegel}} = \arctan \frac{\omega_{\perp}}{\omega_3}$$

$\underline{\omega}$ und $\underline{L}$ (mit $L_j = \Theta_j \omega_j$) rotieren um $\underline{x}_3$
 Flywheelachse

$$L^2 = \Theta^2 \omega_{\perp}^2 + \Theta_3^2 \omega_3^2 \quad (= L_{\perp}^2 + L_3^2)$$

$$\gamma_{\text{Präzessionskegel}} = \arctan \frac{L_{\perp}}{L_3} = \arctan \frac{\Theta \omega_{\perp}}{\Theta_3 \omega_3} = \gamma \text{ (Euler-Winkel)}$$

i.A. $\gamma_{\text{Präzessionskegel}} \neq \gamma_{\text{Polkegel}}$



Bislang: Körperfestes System! S^*

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Nun: Raumfestes System: S

Euler-Winkel:
$$\begin{pmatrix} \omega_1^* \\ \omega_2^* \\ \omega_3^* \end{pmatrix} = \dot{\varphi} \begin{pmatrix} \sin\varphi \sin\psi \\ \cos\varphi \sin\psi \\ \cos\psi \end{pmatrix} + \dot{\psi} \begin{pmatrix} \cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} + \dot{\chi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \dot{\varphi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \dot{\psi} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} + \dot{\chi} \begin{pmatrix} \sin\varphi \sin\psi \\ -\cos\varphi \sin\psi \\ \cos\psi \end{pmatrix}$$

Nun: $\dot{\chi} = 0 \Rightarrow$
$$\begin{pmatrix} \omega_1^* \\ \omega_2^* \\ \omega_3^* \end{pmatrix} = \dot{\varphi} \begin{pmatrix} \sin\varphi \sin\psi \\ \cos\varphi \sin\psi \\ \cos\psi \end{pmatrix} + \dot{\psi} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

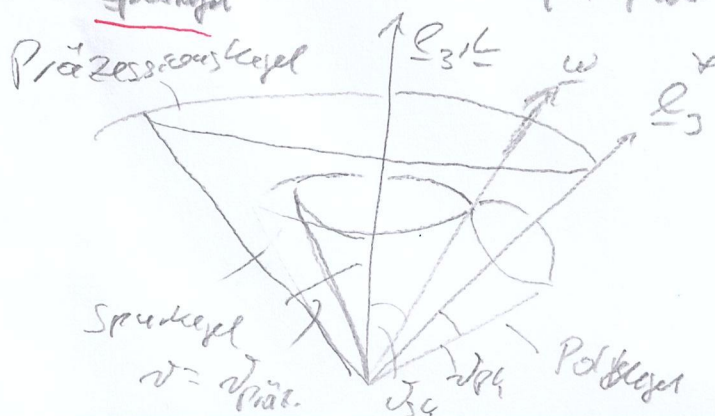
$$= \begin{pmatrix} \omega_3^* \sin(\psi(t)) \\ \omega_3^* \cos(\psi(t)) \\ \cos\psi \end{pmatrix}$$

$$\Rightarrow \varphi = \Omega t \Rightarrow \dot{\varphi} = \Omega = \text{const}$$

$$\dot{\varphi} = \frac{\omega_1^*}{\sin\psi} = \text{const} = \frac{\partial_s \omega_3^*}{\partial_{\cos\psi}} = \Omega' = \frac{L}{\Theta}$$

$\Rightarrow$ Euler-Winkel: $\psi(t) = \psi_0$, $\varphi(t) = \Omega t + \varphi_0$, $\chi(t) = \Omega' t + \chi_0$

Spinnkegel: $\frac{\dot{\varphi} \sin\psi}{\dot{\varphi} + \dot{\psi} \cos\psi} = \text{const} = \frac{\Omega' \sin\psi}{\Omega' + \Omega \cos\psi}$



wann der Erde
Kräftestrahl symm. Kugel:

$$\frac{\Omega - \Omega_3}{\Omega} \approx \frac{1}{300}$$

in Realität: $\frac{1}{430}$ (Bezeichnen...)

6.3 Angular momentum & equations of motion

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Center of mass theorem: $M \ddot{\underline{r}}_0 = \sum_{i=1}^N \underline{\vec{F}}^{(i) \text{ external}} = \underline{\vec{F}}^{\text{external}}$

Angular momentum theorem: $\sum_{i=1}^N m_i \underline{d}^{(i)} \times \ddot{\underline{d}}^{(i)} = \sum_{i=1}^N \underline{d}^{(i)} \times \underline{\vec{F}}^{(i) \text{ external}}$

$$\frac{d}{dt} \underline{L} = \underline{M} \quad (\text{torque})$$

• $\frac{d}{dt} \underline{L} = \frac{d^k}{dt^k} \underline{L} + \underline{\omega} \times \underline{L}$

↑
body-fixed derivative

• Euler's equations: $\frac{d}{dt} \underline{L} = \underline{\Theta} \dot{\underline{\omega}} + \underline{\omega} \times \underline{\Theta} \underline{\omega} = \underline{M}$

• $\Leftrightarrow \begin{cases} \Theta_1 \dot{\omega}_1 + (\Theta_3 - \Theta_2) \omega_2 \omega_3 = M_1 \\ \Theta_2 \dot{\omega}_2 + (\Theta_1 - \Theta_3) \omega_3 \omega_1 = M_2 \\ \Theta_3 \dot{\omega}_3 + (\Theta_2 - \Theta_1) \omega_1 \omega_2 = M_3 \end{cases}$ coupled, nonlinear eqs.

↑
 $L_i = \Theta_i \omega_i$

6.4 Examples:

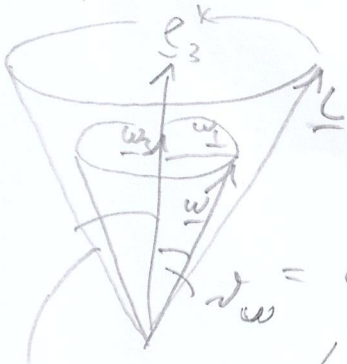
1) Torque-free top (Symmetric: $\Theta_1 = \Theta_2 = \Theta \neq \Theta_3$)

$\Rightarrow \omega_3 = \text{const}$

$\omega_1(t) = \omega_1 \sin(\omega_0 t - \varphi_0)$

$\omega_2(t) = -\omega_1 \cos(\omega_0 t - \varphi_0)$

$\omega_1^2 + \omega_2^2 = \omega_1^2$



$\delta = \arctan \frac{\omega_1}{\omega_3}$ (Kite)

$\delta = \arctan \frac{L_1}{L_3} = \arctan \frac{\Theta \omega_1}{\Theta_3 \omega_3} = \delta$ (Euler angle)

Signal of antenna / Leb focus:

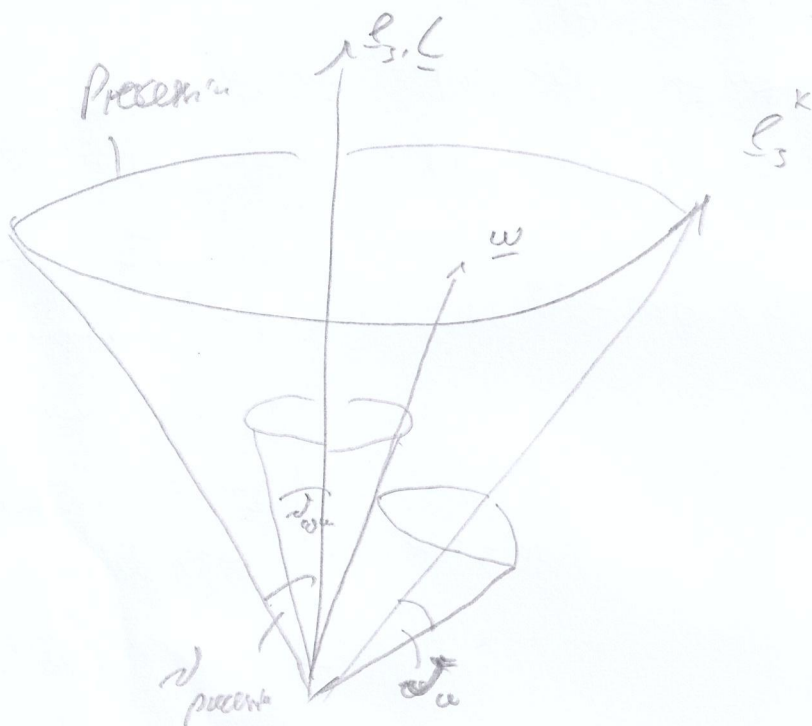
$$\begin{pmatrix} \omega_1^k \\ \omega_2^k \\ \omega_3^k \end{pmatrix} = \varphi \cdot \begin{pmatrix} \sin \varphi \sin \psi \\ -\cos \varphi \sin \psi \\ \cos \psi \end{pmatrix} + \varphi \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \dot{\varphi} = 0$$

$$= \begin{pmatrix} \omega_1^k \sin(\psi(t)) \\ \omega_2^k \cos(\psi(t)) \\ \text{const} \end{pmatrix}$$

→ Euler angles: $\varphi = \Omega t + \varphi_0$, $\dot{\varphi} = \Omega' = \frac{\partial \omega_3^k}{\partial \cos \psi}$, $\psi(t) = \psi_0$

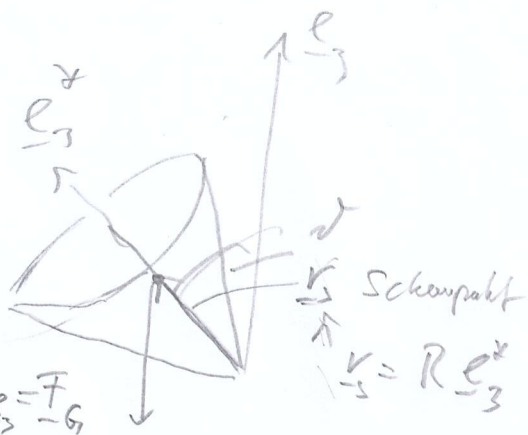
$\dot{\varphi} = \Omega \quad \Rightarrow \quad \varphi = \Omega' t + \varphi_0$

$\angle(\underline{e}_3, \underline{e}_3^k) = \psi_{\text{cone}} = \arccos \frac{\Omega \sin \psi_0}{\Omega' + \Omega \cos \psi}$



2) Schauer, symmetrischer Kreis:

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$$\underline{F}_G = -mg \underline{e}_3 \quad \text{größt am Schwerpunkt}$$

$$\text{Drehmoment: } \underline{M} = \underline{r} \times \underline{F}$$

$$= -mg R \underline{e}_3^* \times \underline{e}_3$$

$$\Rightarrow \underline{M} \perp \text{Figuroachse}$$

$$\text{Gesamtenergie: } E = T + V = \text{const}$$

$$V = mg \underline{e}_3 \cdot \underline{r}_s = mg R \underline{e}_3 \cdot \underline{e}_3^* = mg R \cos \vartheta$$

$$T = \frac{1}{2} \Theta (\omega_1^2 + \omega_2^2) + \frac{1}{2} \Theta_3 \omega_3^2$$

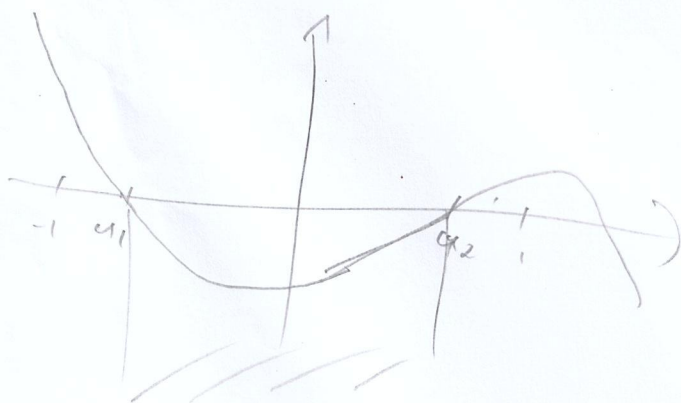
$$\Rightarrow E = \frac{\Theta}{2} (\dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta) + \frac{\Theta_3}{2} (\dot{\varphi} + \dot{\vartheta} \cos \vartheta)^2 + mg R \cos \vartheta = \text{const}$$

$$\hookrightarrow E = \frac{\Theta}{2} \dot{\vartheta}^2 + \frac{(L_3 - L_3^* \cos \vartheta)^2}{2\Theta \sin^2 \vartheta} + \frac{L_3^2}{2\Theta_3} + mg R \cos \vartheta$$

$$\stackrel{E}{\underset{T}{\hookrightarrow}} \quad \dot{\vartheta}^2 + V_{\text{eff}}(\vartheta) = 0, \quad \dot{\vartheta} = -\dot{\varphi} \sin \vartheta$$

$$u = \cos \vartheta$$

$$\begin{aligned} \Rightarrow V_{\text{eff}}(u) &= \frac{2}{\Theta} \left(\frac{L_3^2}{2\Theta_3} + mg R u - E \right) (1 - u^2) + \left(\frac{L_3 - L_3^* u}{\Theta} \right)^2 \\ &= (C_1 + C_2 u) (1 - u^2) + (C_3 - C_4 u)^2 \end{aligned}$$



Bedingung der Figuroachse: $-1 < u_1 \leq \cos \vartheta \leq u_2 < 1$ (Neustadion)

$\Rightarrow$ Dynamik im System!

3) Kinetik freier Massenpunkte

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$$0 < \vartheta_1 < \vartheta_2 < \vartheta_3$$

$$\dot{\omega}_1 = -\frac{\vartheta_3 - \vartheta_2}{\vartheta_1} \omega_2 \omega_3 = -\alpha_1 \omega_2 \omega_3$$

$$\dot{\omega}_2 = \frac{\vartheta_3 - \vartheta_1}{\vartheta_2} \omega_3 \omega_1 = \alpha_2 \omega_3 \omega_1$$

$$\dot{\omega}_3 = -\frac{\vartheta_2 - \vartheta_1}{\vartheta_3} \omega_1 \omega_2 = -\alpha_3 \omega_1 \omega_2$$

Fixpunkte: $\underline{\omega}^* = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$ oder $\begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix}$ oder $\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$
(Rotation um x- y- z-Achse)

Lineareisierung: $f(\underline{\omega}) = \underline{\omega} - \underline{\omega}^*$

$$\begin{pmatrix} f\dot{\omega}_1 \\ f\dot{\omega}_2 \\ f\dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_1 \omega_3 & -\alpha_1 \omega_2 \\ \alpha_2 \omega_3 & 0 & \alpha_2 \omega_1 \\ -\alpha_3 \omega_2 & \alpha_3 \omega_1 & 0 \end{pmatrix} \begin{pmatrix} f\omega_1 \\ f\omega_2 \\ f\omega_3 \end{pmatrix}$$

Eigenwerte für:

$$\underline{\omega}^{(1)} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}: 0 = \det(\underline{A} - \lambda \underline{1}) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & \alpha_2 \omega \\ 0 & \alpha_3 \omega & -\lambda \end{vmatrix} = -\lambda (\lambda^2 + \alpha_2 \alpha_3 \omega^2)$$

$$\Rightarrow \lambda_1^{(1)} = 0, \lambda_{2,3}^{(1)} = \pm i\omega \sqrt{\alpha_2 \alpha_3} \quad \text{Stabil / Zentrum}$$

$$\underline{\omega}^{(2)} = \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix}: 0 = \begin{vmatrix} -\lambda & 0 & \alpha_1 \omega \\ 0 & -\lambda & 0 \\ -\alpha_3 \omega & 0 & -\lambda \end{vmatrix} = -\lambda (\lambda^2 - \alpha_1 \alpha_3 \omega^2)$$

$$\Rightarrow \lambda_1^{(2)} = 0, \lambda_{2,3}^{(2)} = \pm \omega \sqrt{\alpha_1 \alpha_3} \quad \text{instabil / Sattelpunkt}$$

$$\underline{\omega}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}: 0 = \begin{vmatrix} -\lambda & \alpha_1 \omega & 0 \\ \alpha_2 \omega & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda (\lambda^2 + \alpha_1 \alpha_2 \omega^2)$$

$$\Rightarrow \lambda_1^{(3)} = 0, \lambda_{2,3}^{(3)} = \pm i\omega \sqrt{\alpha_1 \alpha_2} \quad \text{Stabil / Zentrum}$$

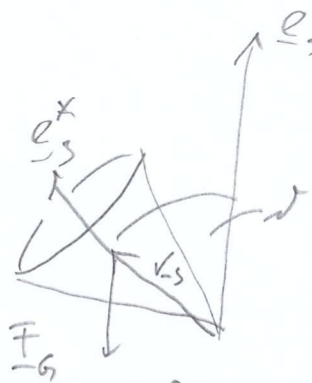
$\Rightarrow$ Dynamik des Systems

6.4 Examples:

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(1) torque-free top (symmetric)

(2) symmetric top subject to gravity



$$\underline{r}_s = R \underline{e}_3^* \text{ axis of symmetry, } \underline{F}_g = -mg \underline{e}_3$$

$$\Rightarrow \text{Torque: } \underline{M} = \underline{r}_s \times \underline{F}_g = -mg R \underline{e}_3^* \times \underline{e}_3$$

$$\Rightarrow \underline{M} \perp \underline{e}_3^* \wedge \underline{M} \perp \underline{e}_3$$

$$\text{total energy: } E = T + V = \frac{1}{2} I (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 + mgR \cos \theta$$

$$= \text{const}$$

$$\hookrightarrow \dot{u}^2 + V_{\text{eff}}(u) = 0$$

motion with $-1 < u_1 \leq \cos \theta \leq u_2 < 1$ Nutation

(3) torque-free asymmetric: $0 < \theta_1 < \theta_2 < \theta_3 < \pi$

$$\left. \begin{aligned} \dot{\omega}_1 &= -d_1 \omega_2 \omega_3 \\ \dot{\omega}_2 &= d_2 \omega_3 \omega_1 \\ \dot{\omega}_3 &= -d_3 \omega_1 \omega_2 \end{aligned} \right\} d_i > 0$$

$$d_1 = \frac{\theta_3 - \theta_2}{\theta_1 - \theta_2}, \quad d_2 = \frac{\theta_3 - \theta_1}{\theta_2 - \theta_1}, \quad d_3 = \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1}$$

$$\text{fixed points: } \underline{\omega}^* \in \left\{ \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \right\}$$

$$\hookrightarrow \text{linearization: } \underline{\dot{\omega}} = \underline{\omega} - \underline{\omega}^*$$

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & -d_1 \omega_3 & -d_1 \omega_2 \\ d_2 \omega_3 & 0 & d_2 \omega_1 \\ -d_3 \omega_2 & -d_3 \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

(not unstable)

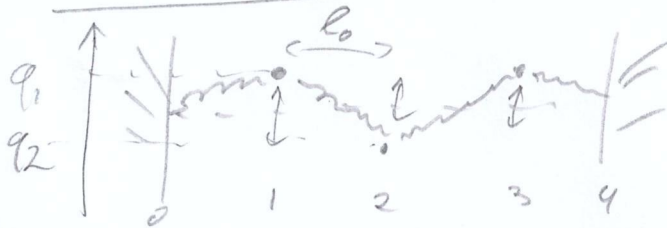
$\Rightarrow$ eigenvalues:

$(\omega, 0, 0)$	$\lambda_1 = 0, \lambda_{2,3} = \pm i\omega \sqrt{d_1 d_3}$	stable / center
$(0, \omega, 0)$	$\lambda_1 = 0, \lambda_{2,3} = \pm i\omega \sqrt{d_1 d_2}$	unstable / saddle
$(0, 0, \omega)$	$\lambda_1 = 0, \lambda_{2,3} = \pm i\omega \sqrt{d_2 d_3}$	stable / center

$\Rightarrow$ 1 unstable motion / rotation around axis relative to θ_2

- 7.1 Kettenmechanik mit 3 Massen / 2 gekoppelte Pendel/Schwingen
 7.2 Übergang zum Kontinuum
 7.3 Wellengleichung

7.1 3 gekoppelte Kettenmechanik



hier: transversale Bewegung

$$L = T - U$$

$$= \frac{m}{2} \sum_{i=0}^{N-1} \dot{q}_i^2 - \frac{D}{2} \sum_{i=0}^{N-1} (q_i - q_{i+1})^2$$

Bewegungsgleichung: (Herleitung über Formales und dann Wahl)

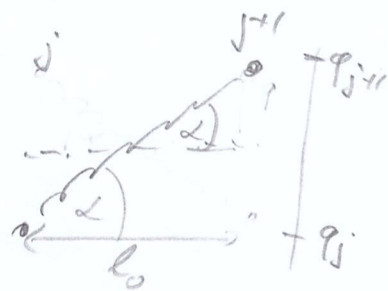
$$m \ddot{q}_j - D (q_{j-1} - 2q_j + q_{j+1}) = 0, \quad j=1, 2, 3$$

Randbedingung $q_0(t) = q_4(t) = 0$

mit F Streck- & F Druck = $F \frac{q_{j+1} - q_j}{l_0}$

$$m \ddot{q}_j + \frac{F}{l_0} (2q_j - q_{j+1} - q_{j-1}) = 0$$

$\downarrow$
 D



Langitudinal & transversale Bewegung haben dieselbe Gleichung!

→ Lösung: $q_j(t) = C q_j \cos(\omega t - \delta)$

$$\Rightarrow (2D - m\omega^2) a_1 - D a_2 = 0$$

$$-D a_1 + (2D - m\omega^2) a_2 - D a_3 = 0$$

$$-D a_2 + (2D - m\omega^2) a_3 = 0$$

⇒ Typ: Eigenwertgleichung (Eigenwerte ω^2 und Eigenvektoren (a_1, a_2, a_3))

$$\begin{pmatrix} 2\frac{D}{m} & -\frac{D}{m} & 0 \\ -\frac{D}{m} & 2\frac{D}{m} & -\frac{D}{m} \\ 0 & -\frac{D}{m} & 2\frac{D}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \frac{D}{m} = \omega^2$$

$$0 = \begin{vmatrix} 2\omega_0^2 - \lambda & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \lambda & -\omega_0^2 \\ 0 & -\omega_0^2 & 2\omega_0^2 - \lambda \end{vmatrix}$$

$$= (2\omega_0^2 - \lambda) \begin{vmatrix} 2\omega_0^2 - \lambda & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \lambda \end{vmatrix} - (\omega_0^4) \begin{vmatrix} -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \lambda \end{vmatrix}$$

$$= (2\omega_0^2 - \lambda) [(2\omega_0^2 - \lambda)^2 - \omega_0^4] + \omega_0^2 [-\omega_0^2 (2\omega_0^2 - \lambda)]$$

$$= (2\omega_0^2 - \lambda) [(2\omega_0^2 - \lambda)^2 - 2\omega_0^4]$$

$$= (2\omega_0^2 - \lambda) [4\omega_0^2 - 4\omega_0^2 \lambda + \lambda^2 - 2\omega_0^4]$$

$$\Rightarrow \lambda_1 = 2\omega_0^2$$

$$\Rightarrow \lambda_{2,3} = 2\omega_0^2 \pm \sqrt{4\omega_0^4 - 2\omega_0^4}$$

$$= 2\omega_0^2 \pm \sqrt{2\omega_0^4} = (2 \pm \sqrt{2})\omega_0^2$$

$$\Rightarrow \lambda_1 = 2\omega_0^2 = 2 \frac{D}{m}$$

$$\lambda_{2,3} = (2 \pm \sqrt{2})\omega_0^2 = (2 \pm \sqrt{2}) \frac{D}{m}$$

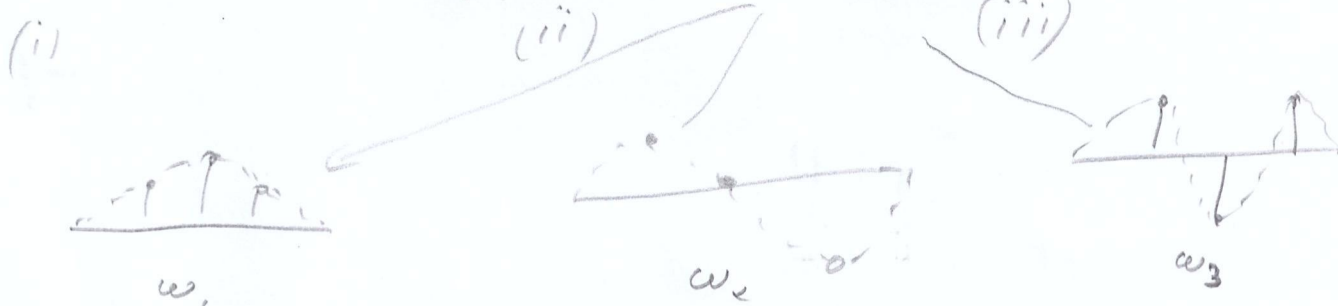
Lösung:

$$(i) \quad \omega_1^2 = (2 - \sqrt{2}) \frac{D}{m}, \quad \underline{a}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$(ii) \quad \omega_2^2 = 2 \frac{D}{m}, \quad \underline{a}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{— Funktional-Schwingungen}$$

$$(iii) \quad \omega_3^2 = (2 + \sqrt{2}) \frac{D}{m}, \quad \underline{a}^{(3)} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Bezugswerte: Norm auf Steuerskurven als Erhaltungswerte (max. Auslenkung)



$$\Rightarrow \text{Lösung: } \underline{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = C_1 \cos(\omega_1 t - \delta_1) \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} + C_2 \cos(\omega_2 t - \delta_2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_3 \cos(\omega_3 t - \delta_3) \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

q ist der Vektor aus Eigenvektoren

7.2 Übergang zum Kontinuum

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Kette: 1, 2, 3, viele, d.h. jetzt N Schwinge

Bewegungsgleichung: $m \ddot{q}_j + D(2q_j - q_{j+1} - q_{j-1}) = 0$, $j=1, \dots, N$

$q_0(t) = 0$, $q_{N+1}(t) = 0$ eingespannt am Rand

Auswahl: $q_j(t) = C q_j \cos(\omega t - j) = C \sin(\alpha j) \cos(\omega t - j)$

mit $\sin((N+1)\alpha) = 0 \Leftrightarrow (N+1)\alpha = \pi k$

$$\hookrightarrow \alpha = \frac{\pi k}{N+1} \quad k=1, 2, \dots$$

$$\Rightarrow \underbrace{[-m\omega^2 + 2D(1 - \cos(\frac{\pi k}{N+1}))]}_{=0} C \sin(\alpha j) = 0$$

$$\text{wegen } D(2q_j - q_{j+1} - q_{j-1}) = C \cos(\omega t - j) D [\sin(\alpha j) - \sin(\alpha(j+1)) + \sin(\alpha j) - \sin(\alpha(j-1))]$$

(Trick: komplexe Darstellung von $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$)

$$= 2D \cos(\omega t - j) (1 - \cos \alpha) \sin(\alpha j)$$

keine triviale Lösung für $C \neq 0$, $\sin(\alpha j) \neq 0$:

$$\omega^2 = \frac{2D}{m} (1 - \cos \frac{\pi k}{N+1}) \stackrel{2 \sin^2 x = 1 - \cos(2x)}{=} \frac{4D}{m} \sin^2 \frac{\pi k}{2(N+1)}$$

$$\Rightarrow \omega_k = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi k}{2(N+1)}, \quad k=1, \dots, N$$

$$N=1: \omega_1 = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi \cdot 1}{2 \cdot 2} = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi}{4} = 2 \sqrt{\frac{D}{m}} \frac{1}{2} \sqrt{2} = \sqrt{2} \omega_0$$

$$N=2: \omega_2 = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi \cdot 1}{2 \cdot 3} \Rightarrow k=1: \omega_{2,1} = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi}{6} = \sqrt{\frac{D}{m}} = \omega_0$$

$$k=2: \omega_{2,2} = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi}{3} = 2 \sqrt{\frac{D}{m}} \frac{1}{2} \sqrt{3} = \sqrt{3} \omega_0$$

$$N=3: \omega_3 = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi}{8} \Rightarrow k=1: \omega_{3,1} = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi}{8} = 2 \sqrt{\frac{D}{m}} \frac{1}{2} \sqrt{2} = \sqrt{2} \omega_0$$

$$k=2: \omega_{3,2} = 2 \sqrt{\frac{D}{m}} \sin \frac{\pi}{4} = 2 \sqrt{\frac{D}{m}} \frac{1}{2} \sqrt{2} = \sqrt{2} \omega_0$$

$$k=3: \omega_{3,3} = 2 \sqrt{\frac{D}{m}} \sin \frac{3\pi}{8} = 2 \sqrt{\frac{D}{m}} \frac{1}{2} (1 + \sqrt{2}) = (1 + \sqrt{2}) \omega_0$$

$$\sin \frac{\pi}{8} = \frac{1}{2} \sqrt{2 - \sqrt{2}}$$

Nebeuvende

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$$\sin(\alpha j) = \frac{1}{2i} (e^{i\alpha j} - e^{-i\alpha j})$$

$$\Rightarrow 2\sin(\alpha j) - 2\sin(\alpha(j+1)) - 2\sin(\alpha(j-1))$$

$$= \frac{1}{2i} (2e^{i\alpha j} - 2e^{-i\alpha(j+1)} - 2e^{i\alpha j} e^{i\alpha} + e^{-i\alpha(j-1)} e^{-i\alpha} - 2e^{i\alpha(j-1)} + e^{-i\alpha(j-1)} e^{-i\alpha})$$

$$= \frac{1}{2i} \left[e^{i\alpha j} (2 - e^{i\alpha} - e^{-i\alpha}) - e^{-i\alpha j} (2 - e^{-i\alpha} - e^{i\alpha}) \right]$$

$$= \frac{1}{2i} (2 - e^{i\alpha} - e^{-i\alpha}) (e^{i\alpha j} - e^{-i\alpha j})$$

$$= 2 \left[2 - \underbrace{(e^{i\alpha} + e^{-i\alpha})}_{2\cos\alpha} \right] \sin(\alpha j)$$

$$= 2(1 - \cos\alpha) \sin(\alpha j)$$

$$\Rightarrow q_j = q_{jk} = \sin\left(\frac{\pi k}{N+1} j\right) \quad j, k = 1, \dots, N$$

$$\Rightarrow q_j(t) = \sum_{k=1}^N C_k \sin\left(\frac{\pi k}{N+1} j\right) \cos(\omega_k t - \delta_k)$$

$$= \sum_{k=1}^N \sin\left(\frac{\pi k}{N+1} j\right) \left(d_k \cos(\omega_k t) + e_k \sin(\omega_k t) \right)$$

Integrationskonstanten

$$\left(\sin\left(\frac{\pi k}{N+1} j\right) \right)$$

$$d_k = \frac{2}{N+1} \sum_{j=1}^N q_j(0) \sin\left(\frac{\pi k}{N+1} j\right), \quad e_k = \frac{2}{N+1} \frac{1}{\omega_k} \sum_{j=1}^N \dot{q}_j(0) \sin\left(\frac{\pi k}{N+1} j\right)$$

6 Fourier-Zerlegung

Übergang zum Kontinuum:

$$N \rightarrow \infty \text{ und } l_0 \rightarrow 0 : (N+1)l_0 = l = \text{const (Gesamtlänge)}$$

$$m \rightarrow 0 \text{ und } l_0 \rightarrow 0 : \frac{m}{l_0} = \rho = \text{const (Massendichte)}$$

$$D \rightarrow \infty \text{ und } l_0 \rightarrow 0 : D l_0 = \text{const. (Potenzielle Dichte)}$$

Interpretation:

$m \square m \rightarrow m \square m \square m \rightarrow \dots$

7.3 Wellengleichung

$\Rightarrow$ Bewegungsgleichung:

$$\frac{m}{l_0} \frac{1}{D l_0} \ddot{\varphi}_j = \frac{1}{l_0} \left[\frac{\varphi_{j+1} - \varphi_j}{l_0} - \frac{\varphi_j - \varphi_{j-1}}{l_0} \right]$$

Nimm: Lim $l_0 \rightarrow 0 : \varphi_j(t) \rightarrow \varphi(x, t)$ mit $x = j l_0$

$$\Rightarrow \frac{m}{l_0} \frac{1}{D l_0} \frac{\partial^2}{\partial t^2} \varphi(x, t) = \lim_{l_0 \rightarrow 0} \frac{1}{l_0} \left[\frac{\varphi(x+l_0, t) - \varphi(x, t)}{l_0} - \frac{\varphi(x, t) - \varphi(x-l_0, t)}{l_0} \right]$$

$$= \lim_{l_0 \rightarrow 0} \frac{1}{l_0} \left(\left. \frac{\partial \varphi}{\partial x} \right|_{x+l_0} - \left. \frac{\partial \varphi}{\partial x} \right|_{x-l_0} \right)$$

$$= \frac{\partial^2 \varphi}{\partial x^2} \Rightarrow \frac{1}{c^2} \frac{\partial^2 \varphi(x, t)}{\partial t^2} = \frac{\partial^2 \varphi(x, t)}{\partial x^2} \text{ mit } \frac{1}{c^2} = \frac{m}{l_0 D l_0}$$

Wellengleichung

7. Continuum mechanics

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7.1 3 spring coupled oscillators



$$L = T - V$$

$$= \frac{m}{2} \sum_{i=0}^{N+1} \dot{q}_i^2 - \frac{D}{2} \sum_{i=0}^{N+1} (q_i - q_{i-1})^2$$

equation of motion: $m \ddot{q}_j = D(q_{j-1} - 2q_j + q_{j+1}) = 0 \quad j=1,2,3$

boundary conditions: $q_0(t) = 0 = q_{N+1}(t)$

solution / ansatz: $q_j(t) = C q_j \cos(\omega t - \delta)$

$$\begin{pmatrix} 2\frac{D}{m} & -\frac{D}{m} & 0 \\ -\frac{D}{m} & 2\frac{D}{m} & -\frac{D}{m} \\ 0 & -\frac{D}{m} & 2\frac{D}{m} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \omega^2 \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \text{eigenvalue equation}$$

$$\Rightarrow \lambda_1 = \omega_1^2 = 2\frac{D}{m} \Rightarrow \omega_1 = \sqrt{2}\omega_0, \quad \omega_0 = \sqrt{\frac{D}{m}}, \quad \underline{a}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_{2,3} = \omega_{2,3}^2 = (2 \pm \sqrt{2})\frac{D}{m} \Rightarrow \omega_2 = \sqrt{2 - \sqrt{2}}\omega_0, \quad \underline{a}^{(2)} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

$$\omega_3 = \sqrt{2 + \sqrt{2}}\omega_0, \quad \underline{a}^{(3)} = \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

7.2 limit to continuum

$$q_j(t) = C \sin(\alpha j) \cos(\omega t - \delta), \quad j=1, \dots, N, \quad q_0(t) = 0 = q_{N+1}(t)$$

with $\alpha = \frac{\pi k}{N+1}$

$$\left[(-m\omega^2 + 2D) \left(1 - \cos\left(\frac{\pi k}{N+1}\right) \right) \right] C \sin(\alpha j) = 0$$

$$\omega_k = 2\sqrt{\frac{D}{m}} \sin\left(\frac{\pi k}{2(N+1)}\right) \quad \text{checked for } k=1,2,3$$

$$\Rightarrow q_j(t) = \sum_{k=1}^N \sin\left(\frac{\pi k}{N+1} j\right) \underbrace{C_k \cos(\omega_k t - \delta_k)}_{[d_k \cos(\omega_k t) + e_k \sin(\omega_k t)]}$$

on sine curves!

limits:

$$N \rightarrow \infty, l_0 \rightarrow 0 : (N+1)l_0 = l = \text{const} \quad (\text{total length})$$

$$m \rightarrow 0, l_0 \rightarrow 0 : \frac{m}{l_0} = \rho = \text{const} \quad (\text{density of mass})$$

$$D \rightarrow \infty, l_0 \rightarrow 0 : D l_0 = \text{const} \quad \begin{array}{l} \text{(total potential energy)} \\ \text{(modulus of elasticity)} \end{array}$$

$$\text{with spatial coordinate: } x = j l_0 \rightarrow q_j(t) \rightarrow \varphi(x, t)$$

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi(x, t) = \frac{\partial^2}{\partial x^2} \varphi(x, t) \quad \text{Wave equation}$$

$$\frac{1}{c^2} = \frac{m}{l_0} \frac{1}{D l_0}$$

$$\Rightarrow a_j = a_{j,k} = \sin\left(\frac{\pi k}{N+1} j\right) \quad j, k = 1, \dots, N$$

$$\Rightarrow q_j(t) = \sum_{k=1}^N C_k \sin\left(\frac{\pi k}{N+1} j\right) \cos(\omega_k t - \delta_k)$$

$$= \sum_{k=1}^N \sin\left(\frac{\pi k}{N+1} j\right) \left(d_k \cos(\omega_k t) + e_k \sin(\omega_k t) \right)$$

Integrationskonstanten

$$d_k = \frac{2}{N+1} \sum_{j=1}^N q_j(0) \sin\left(\frac{\pi k}{N+1} j\right), \quad e_k = \frac{2}{N+1} \frac{1}{\omega_k} \sum_{j=1}^N \dot{q}_j(0) \sin\left(\frac{\pi k}{N+1} j\right)$$

↳ Fourier-Zerlegung

Übergang zum Kontinuum:

$$N \rightarrow \infty \text{ und } l_0 \rightarrow 0 : (N+1)l_0 = l = \text{const (Gesamtlänge)}$$

$$m \rightarrow 0 \text{ und } l_0 \rightarrow 0 : \frac{m}{l_0} = \rho = \text{const (Massendichte)}$$

$$D \rightarrow \infty \text{ und } l_0 \rightarrow 0 : D l_0 = \text{const.} \quad \begin{matrix} = \gamma = D \\ \text{(Potenzielle Energie)} \\ \text{(Elastizitätsmodul)} \end{matrix}$$

Interferenz:

mm □ mm

→

mm □ mm

→

7.3 Wellengleichung

⇒ Bewegungsgleichung:

$$\frac{m}{l_0} \frac{1}{D l_0} \ddot{\varphi}_j = \frac{1}{l_0} \left[\frac{\varphi_{j+1} - \varphi_j}{l_0} - \frac{\varphi_j - \varphi_{j-1}}{l_0} \right]$$

Nimm: Lim $l_0 \rightarrow 0$: $\varphi_j(t) \rightarrow \varphi(x, t)$ mit $x = j l_0$

$$\Rightarrow \frac{m}{l_0} \frac{1}{D l_0} \frac{\partial^2}{\partial t^2} \varphi(x, t) = \lim_{l_0 \rightarrow 0} \frac{1}{l_0} \left[\frac{\varphi(x+l_0, t) - \varphi(x, t)}{l_0} - \frac{\varphi(x, t) - \varphi(x-l_0, t)}{l_0} \right]$$

$$= \lim_{l_0 \rightarrow 0} \frac{1}{l_0} \left(\frac{\partial \varphi}{\partial x} \Big|_{x+l_0} - \frac{\partial \varphi}{\partial x} \Big|_{x-l_0} \right)$$

$$= \frac{\partial^2 \varphi}{\partial x^2} \Rightarrow \frac{1}{c^2} \frac{\partial^2 \varphi(x, t)}{\partial t^2} = \frac{\partial^2 \varphi(x, t)}{\partial x^2} \text{ mit } \frac{1}{c^2} = \frac{m}{l_0 D}$$

Wellengleichung

mit $\frac{m}{l_0} \frac{1}{Dl_0} = \frac{g}{Dl_0} =: \frac{1}{c^2}$:

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$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(x,t) = \frac{\partial^2}{\partial x^2} \psi(x,t) \quad \text{Wellengleichung}$$

$$[c^2] = \left[\frac{Dl_0}{g} \right] = \left[\frac{\frac{F}{l_0} l_0}{g} \right] = \left[\frac{F l_0}{g} \right] = \frac{\frac{kg \frac{m}{s^2}}{m}}{\frac{kg}{m}} = \frac{m^2}{s^2} \Rightarrow [c] = \frac{m}{s}$$

c : Phasengeschwindigkeit

Lösungen:

(i) D'Alembertsche Lösung: $\psi(x,t) = f(x \pm ct)$

im mit bewegtem Koordinatensystem: stehende Welle

(ii) Bernoullische Lösung: $\psi(x,t) = g(x) h(t)$

$$\Rightarrow \frac{1}{c^2} g(x) \ddot{h}(t) = g''(x) h(t) \Rightarrow \frac{1}{c^2} \frac{\ddot{h}(t)}{h(t)} = \frac{g''(x)}{g(x)} = -k^2 = \text{const.}$$

Also: $\ddot{h}(t) = -c^2 k^2 h(t)$ und $g''(x) = -k^2 g(x)$

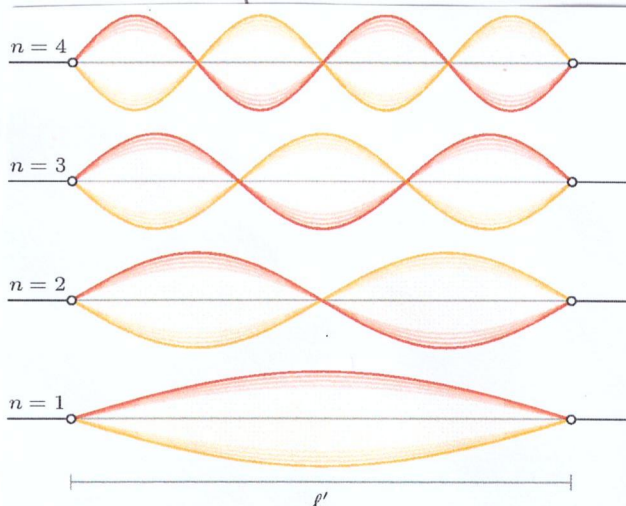
$\Rightarrow h(t) = h_1 \cos(kc t) + h_2 \sin(kc t)$ und $g(x) = g_1 \cos(kx) + g_2 \sin(kx)$

Randbedingung des festgehaltenen, schwingenden Kleideins (unendliche Kette):

$g(0) = g(l) = 0 \Rightarrow g_1 = 0, \sin(kl) = 0 \Rightarrow kl = n\pi \quad n=1,2,\dots$

mit $k_n c t = \omega_n t$: Eigenfrequenzen $k_n = \frac{n\pi}{l}$ $\omega_n = n \frac{\pi c}{l}$

$\Rightarrow \psi_n(x,t) = \sin \frac{n\pi x}{l} \left(d_n \cos(\omega_n t) + e_n \sin(\omega_n t) \right)$ Eigen Schwingung



1. Harmonische

3. Kosinusmode

2. Kosinusmode

Grundschwingung

⇒ allgemeine Lösung:

$$\varphi(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(d_n \cos(\omega_n t) + e_n \sin(\omega_n t) \right)$$

$$d_n = \frac{2}{l} \int_0^l \varphi(x, 0) \sin \frac{n\pi x}{l} dx \quad e_n = \frac{2}{l\omega_n} \int_0^l \dot{\varphi}(x, 0) \sin \frac{n\pi x}{l} dx$$

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Bsp.: Membran Schwingungen

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \quad \text{mit } \varphi = \varphi(x, y, t)$$

Randbedingung einer rechteckigen Membran



$$x=0, x=a, y=0, y=b$$

$$\varphi(0, y, t) = \varphi(a, y, t) = \varphi(x, 0, t) = \varphi(x, b, t) = 0 \quad \forall t \quad (\text{fest eingespannt})$$

$$\text{Produktansatz: } \varphi(x, y, t) = \sin(kx) \sin(\hat{k}y) \left[d \cos(\omega t) + e \sin(\omega t) \right]$$

$$\text{mit } k = \frac{n\pi}{a}, \quad \hat{k} = \frac{s\pi}{b}, \quad n, s = 1, 2, \dots$$

$$\hookrightarrow \text{Eigenfrequenzen: } \omega_{n,s} = \pi c \sqrt{\frac{n^2}{a^2} + \frac{s^2}{b^2}} \quad \text{irrationale Vielfache von } \omega_{1,1} = \pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

allgemeine Lösung:

$$\varphi(x, y, t) = \sum_{n,s=1}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{s\pi y}{b} \left[d_{n,s} \cos(\omega_{n,s} t) + e_{n,s} \sin(\omega_{n,s} t) \right]$$

mit Fourier Koeffizienten:

$$d_{n,s} = \frac{4}{ab} \int_0^a \int_0^b \varphi(x, y, 0) \sin \frac{n\pi x}{a} \sin \frac{s\pi y}{b} dx dy$$

$$e_{n,s} = \frac{4}{ab \omega_{n,s}} \int_0^a \int_0^b \dot{\varphi}(x, y, 0) \sin \frac{n\pi x}{a} \sin \frac{s\pi y}{b} dx dy$$

Frequenzen (i) i.A. irrationale Vielfache ⇒ kein harmonisches Obertöne

$$\text{etwa: für } a=b: \omega_{1,2} = \pi c \sqrt{\frac{5}{a^2}} = \sqrt{5} \omega_{1,1}$$

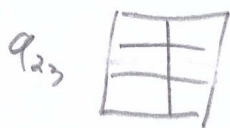
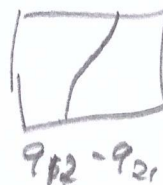
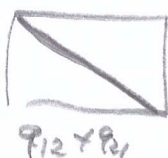
(ii) Entartung: Schwingungen mit gleicher Eigenfrequenz $\omega_{n_1,s_1} = \omega_{n_2,s_2}$:

$$\frac{n_1^2}{a^2} + \frac{s_1^2}{b^2} = \frac{n_2^2}{a^2} + \frac{s_2^2}{b^2} \Leftrightarrow \frac{a^2}{b^2} = \frac{n_1^2 - n_2^2}{s_2^2 - s_1^2} \Rightarrow \text{Bed. } \frac{a^2}{b^2} \in \mathbb{Q}$$

Knotenlinien: $\psi(x, y, t) = 0$

(a) parallel zum Rand: $x = 0, \frac{a}{n}, \frac{2a}{n}, \dots, a$

$$y = 0, \frac{b}{s}, \frac{2b}{s}, \dots, b$$



(b) $a = b$, $d_{12}/d_{21} = q_{12}/q_{21} = u$, $\omega_{12} = \omega_{21} = \bar{\omega}$, *Auswertung von der Eigenschwingung q_{12}, q_{21}*

$$\psi(x, y, t) = \left[u \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} + \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right]$$

$$(d_{21} \cos(\bar{\omega} t) + e_{21} \sin(\bar{\omega} t))$$

$$\Rightarrow [\dots] = 0 \Leftrightarrow u \cos \frac{\pi y}{a} + \cos \frac{\pi y}{a} = 0$$

$$\uparrow \sin(2\pi/2) = 2 \sin \pi \cos \pi$$

$$\left(u \sin \frac{\pi x}{a} \cdot 2 \sin \frac{\pi y}{a} \cos \frac{\pi y}{a} + \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \sin \frac{\pi y}{a} \right)$$

$$= 2 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \left(u \cos \frac{\pi y}{a} + \cos \frac{\pi y}{a} \right)$$

$u = \pm 1$: Diagonale

(c) kreisförmige Membran: $\rightarrow$ Polarkoordinaten (r, φ)

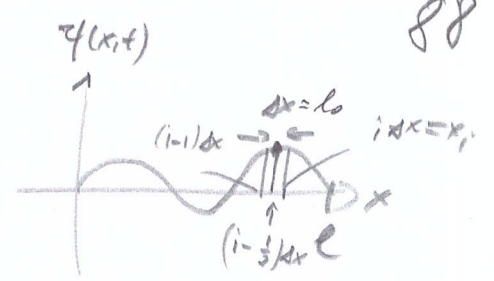
Besselsche Gleichung: $\psi(r, \varphi) = R(r) \phi(\varphi)$

$$\hookrightarrow \phi'' + \lambda^2 \phi = 0 \quad \text{Winkelanteil}$$

$$r^2 R'' + r R' + (k^2 r^2 - \lambda^2) R = 0 \quad \text{Radialanteil (Bessel-DGL)}$$

Linear der Lagrange-Funktion:

$$T = \sum_{i=1}^N \frac{m_i}{2} \left(\frac{dq_i}{dt} \right)^2 = \sum_{i=1}^N \frac{\rho \Delta x}{2} \dot{q}_i^2$$



m_i = Masse bei $x_i = (i - \frac{1}{2}) \Delta x$ $i = 1, \dots, N$, ρ : Massendichte $\frac{m}{\Delta x} = \frac{m}{l_0}$

$$T = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{\rho \Delta x}{2} \dot{q}_i^2 = \frac{\rho}{2} \lim_{N \rightarrow \infty} \sum_{i=1}^N \Delta x \left(\frac{\partial y(x_i, t)}{\partial t} \right)^2 = \frac{\rho}{2} \int_0^l dx \left(\frac{\partial y(x, t)}{\partial t} \right)^2$$

Für V : $\lim_{\Delta x \rightarrow 0} \frac{q_{i+1}(t) - q_i(t)}{\Delta x} = \frac{\partial y(x, t)}{\partial x}$

$$\Rightarrow V = \lim_{N \rightarrow \infty} \frac{D}{2} \sum_{i=1}^{N-1} (q_{i+1} - q_i)^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^{N-1} \frac{D(\Delta x)^2}{2} \left(\frac{y(x_{i+1}, t) - y(x_i, t)}{(\Delta x)^2} \right)^2$$

geometrisch: $\Delta s = \sqrt{(\Delta x)^2 + (y(x_{i+1}, t) - y(x_i, t))^2}$
 $\uparrow$
 Abstand $i \leftrightarrow i+1$ $\approx \Delta x \left[1 + \frac{(y(x_{i+1}, t) - y(x_i, t))^2}{2(\Delta x)^2} \right]$

Kraft proportional zu $\Delta s - \Delta x$

Verspannung aufteilen $\Rightarrow V = u - \frac{1}{2} \rho u (\Delta s - \Delta x)^2$

$\downarrow$
 $P = D \Delta x = \text{const}$

$\downarrow$
 $\frac{1}{2} \frac{D \Delta x}{2} \int_0^l dx \left(\frac{\partial y(x, t)}{\partial x} \right)^2$
 $\uparrow$
 $\frac{1}{2} \frac{D \Delta x}{2} \rightarrow \frac{1}{2} \frac{D}{2} \int_0^l dx$

$P = D \Delta x = D l_0$: Verspannung
 elastische Feder

$$\Rightarrow L = T - V = \int_0^l dx \left[\frac{\rho}{2} \left(\frac{\partial y(x, t)}{\partial t} \right)^2 - \frac{D}{2} \left(\frac{\partial y(x, t)}{\partial x} \right)^2 \right]$$

$L(y, y')$ Lagrange-Dichte

• Hamiltonsches Prinzip:

$$\delta S = \delta \int_{t_1}^{t_2} dt \int_0^l dx L(y, \dot{y}, y', x, t) = 0$$

$$\delta L = \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} + \frac{\partial L}{\partial y'} \delta y'$$

• Variations: vgl. 3.2 Nach dem Prinzip der kleinsten Wirkung

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$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}'} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Partielle DGL!
(vgl. $\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0, j=1, \dots, N$)

mit: $\mathcal{L} = (g \dot{\varphi}^2 - D \varphi'^2)/2$ folgt:

$$g \frac{\partial^2 \varphi}{\partial t^2} - D \frac{\partial^2 \varphi}{\partial x^2} = 0$$

Wellengleichung (frei Schwingen)

$$\underbrace{\frac{g}{D}}_{\frac{1}{c^2}} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} \quad \left(+ \varphi f(x,t) \right)$$

äußere Kraftdichte

(Empirisch: $\Delta V = -\varphi f \Delta x$)

etwa: $f = -g \left(\frac{g}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{D}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + f(x,t) \varphi \right)$

Typ: Wellengleichung für Feld φ (Feldgleichung)

$= T - V$
kin. Abstraktion

• 2/30: $\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)}_{\Delta} \varphi(x,y,z,t)$

Δ : D'Alembert-Operator

7.3 Wave equation

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$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(x,t) = \frac{\partial^2}{\partial x^2} \psi(x,t) \quad c: (\text{phase}) \text{ velocity}$$

↳ D'Alembert / ^{ansatz} solution $\psi(x,t) = f(x \pm ct)$

↳ standing waves in co-moving frame

↳ Bernoulli / ^{ansatz} solution: $\psi(x,t) = g(x) h(t)$

$$\Rightarrow h''(t) = -c^2 k^2 h(t) \quad \text{and} \quad g''(x) = -k^2 g(x)$$

e.g. 1) guitar string:

$$\psi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l} x\right) \left(d_n \cos(\omega_n t) + e_n \sin(\omega_n t) \right)$$

(ii) 2D rectangular membrane: $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(x,y,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x,y,t)$

↳ $\psi(x,y,t) = \sum_{n,m=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \left[d_{nm} \cos(\omega_{nm} t) + e_{nm} \sin(\omega_{nm} t) \right]$

$$\omega_{nm} = \pi c \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \quad + f(x,y) \psi(x,y)$$

• Lact of Lagrange function:

$$L = T - V = \int dx \left[\frac{\rho}{2} \left(\frac{\partial \psi(x,t)}{\partial t} \right)^2 - \frac{\rho}{2} \left(\frac{\partial \psi}{\partial x} \right)^2 \right]$$

L : Lagrange density

$$\delta S = \int_{t_1}^{t_2} dx \delta L = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}} + \frac{\partial}{\partial x} \frac{\partial L}{\partial \psi'} - \frac{\partial L}{\partial \psi} = 0 \quad \left(\text{if } \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \right)$$

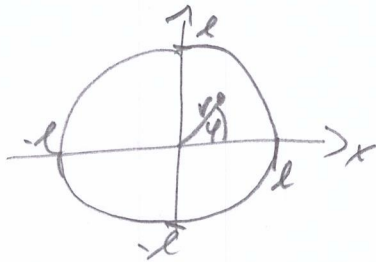
$\psi(x,t)$ in case of external force

$$\text{total force } \tilde{F}(t) = \int_l dx f(x,t)$$

Start: Wellengleichung (2D): $u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$

$$\text{Laplace-Op.} \rightarrow \Delta u(x,y,t) = \frac{1}{c^2} \frac{\partial^2 u(x,y,t)}{\partial t^2}$$

mit Randbedingung: $u(r=l, \varphi, t) = 0$ in Polarkoordinaten



$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

Auflagebedingung: $u(r, \varphi, 0)$ und $\frac{\partial}{\partial t} u(r, \varphi, 0)$

Idee: 2x Separationsansatz + Rekursionsformel

$$u(r, \varphi, t) = f(r, \varphi) T(t)$$

$$\Rightarrow \frac{\Delta f(r, \varphi)}{f(r, \varphi)} = \frac{1}{c^2} \frac{\ddot{T}(t)}{T(t)} = -k^2 = \text{const.}$$

$$\Rightarrow T(t) = A \cos(\omega t) + B \sin(\omega t), \quad \omega = k c$$

$$\Rightarrow \text{Otzenteil: } \Delta f(r, \varphi) + k^2 f(r, \varphi) = 0 \text{ mit } f(l, \varphi) = 0$$

in Polarkoordinaten

$$\Delta f(r, \varphi) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} = \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2}$$

$$\Rightarrow \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + k^2 f = 0$$

Separationsansatz: $f(r, \varphi) = R(r) \Phi(\varphi)$

$$\frac{\partial^2 \Phi}{\partial \varphi^2} \frac{R}{r^2} + \Phi \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + k^2 R \right) = 0$$

$$\frac{1}{\phi} \frac{\partial^2 \phi}{\partial \varphi^2} = -\frac{1}{R} \left(r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + k^2 r^2 R \right) = -\lambda^2 = \text{const} \quad 97$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial \varphi^2} + \lambda^2 \phi = 0, \quad r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + (k^2 r^2 - \lambda^2) R = 0$$

Winkelanteil

Radialanteil

$$\hookrightarrow \phi(\varphi) = \cos(\lambda \varphi + \varphi_0) \stackrel{!}{=} \phi(\varphi + 2\pi) \text{ Periodizität } \varphi$$

$$\Rightarrow \cos(\lambda(\varphi + 2\pi) + \varphi_0) \stackrel{!}{=} \cos(\lambda \varphi + \varphi_0 + \underbrace{2\pi \lambda})$$

$$\Rightarrow \lambda = m \in \mathbb{N}$$

bleibt noch: $r^2 R'' + r R' + (k^2 r^2 - \lambda^2) R = 0$

mit $x := kr \Rightarrow R' = k \frac{dR}{dx}, \quad R'' = k^2 \frac{d^2 R}{dx^2}$

$$x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + (x^2 - \lambda^2) R = 0 \quad \text{Besselsche DGL}$$

Lösungen: $R(x) = x^p \sum_{s=0}^{\infty} \alpha_s x^s$ (vgl. TP 2a Kap 7.3 (a))

$$\Rightarrow \frac{dR}{dx} = \sum_{s=0}^{\infty} \alpha_s (p+s) x^{(p+s)-1}$$

$$\frac{d^2 R}{dx^2} = \sum_{s=0}^{\infty} \alpha_s (p+s)(p+s-1) x^{(p+s)-2}$$

Einsetzen:

$$\sum_{s=0}^{\infty} \alpha_s \underbrace{[(p+s)(p+s-1) + (p+s) - \lambda^2]}_{(p+s)^2} x^{p+s} + \sum_{s=0}^{\infty} \alpha_s x^{p+s+2} = 0$$

$$\Rightarrow \alpha_0 [p^2 - \lambda^2] x^p + \alpha_1 [(p+1)^2 - \lambda^2] x^{p+1} + \sum_{s=2}^{\infty} \{ \alpha_s [(p+s)^2 - \lambda^2] + \alpha_{s-2} \} x^{p+s} = 0$$

Koeffizientenvergleich:

$$x^p: \alpha_0 [p^2 - \lambda^2] = 0$$

$$\Rightarrow p = \pm \lambda \quad (\lambda \neq 0)$$

$$x^{p+1}: \alpha_1 [(p+1)^2 - \lambda^2] = 0$$

$$\Rightarrow \alpha_1 [(\pm \lambda + 1)^2 - \lambda^2] = 0 \Rightarrow \alpha_1 = 0$$

$$x^{p+s}: \alpha_s [(p+s)^2 - \lambda^2] + \alpha_{s-2} = 0 \quad (s \geq 2)$$

$$\Rightarrow \alpha_s = \frac{\alpha_{s-2}}{(s-2)(s-1-\lambda^2)} = \frac{\alpha_{s-2}}{s(s-1-\lambda^2)}$$

Rekursionsformel

wegen $a_1 = 0$ und Rekursionsformel: alle ungeraden $a_i = a_{2n+1} = 0$ § 2

$$s = +\lambda: a_2 = -\frac{a_0}{2(2\lambda+2)}, a_4 = -\frac{a_2}{4(2\lambda+4)} = \frac{a_0}{2 \cdot 4(2\lambda+2)(2\lambda+4)}$$

$$a_6 = -\frac{a_4}{2 \cdot 4 \cdot 6(2\lambda+2)(2\lambda+4)(2\lambda+6)} \dots$$

mit Fakultät: $a_{2n} = \frac{1}{2^n n!}$ selbst $\lambda = m$

$$J_m(x) = x^m \sum_{s=0}^{\infty} a_s x^s = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(m+r)!} \left(\frac{x}{2}\right)^{m+2r} \quad \text{Bessel-Funktion}$$

$$\Gamma s = -\lambda: J_{-\lambda}(x) = x^{-\lambda} \sum_{s=0}^{\infty} a_s x^s$$

$$\text{Schluss: } \lambda = m \Rightarrow \frac{d^{s-2}}{ds^2} \frac{1}{s! 2^{m+s}}$$

s beginnt bei $s=2m$

$$(\lambda! = \Gamma(\lambda+1))$$

$$\Rightarrow R(r) = D \sum_{k=r}^{\infty} J_m(kr),$$

$$\Rightarrow u(r, \varphi, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} u_{nm}$$

$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[A_{nm} \cos(\omega_{nm} t) + B_{nm} \sin(\omega_{nm} t) \right]$$

$$\cos(m\varphi + \varphi_0) J_m(u_{nm} r)$$

Knotenlinien:



unverändertes Verhältnis

Eigenschaften:

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$$

$$\frac{dJ_m(x)}{dx} = -\frac{m}{x} J_m(x) + J_{m-1}(x)$$

$$\text{Randbedingung: } J_m(\mu_{nm}) = 0$$

$$\Rightarrow \omega_{nm} = \mu_{nm} = \frac{\omega}{c} l$$

$$\Rightarrow \omega_{nm} = \frac{c}{l} \mu_{nm}$$

Erweiterung der Lagrange-Formale auf 3 Raumdimensionen: $x \rightarrow \underline{x}$

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$$q \rightarrow \underline{q} = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}$$

$$L = \int_V d^3x \left[\frac{1}{2} \dot{\underline{q}}^2(\underline{x}, t) - V(\underline{q}, \nabla \underline{q}) \right]$$

mit potenzieller Energiedichte $V(\underline{q}, \nabla \underline{q}) = V^{\text{außen}}(\underline{q}, \underline{x}) + V^{\text{innen}}(\nabla \underline{q})$

$$V^{\text{außen}}(\underline{q}, \underline{x}) = - \int (\underline{x}, \underline{r}) \cdot \underline{q}(\underline{x}, t)$$

$$\nabla \underline{q} = \underline{\eta} = \begin{pmatrix} \partial_x q_x & \partial_x q_y & \partial_x q_z \\ \partial_y q_x & \partial_y q_y & \partial_y q_z \\ \partial_z q_x & \partial_z q_y & \partial_z q_z \end{pmatrix}$$

↑
dyadisches
Produkt

$$\Rightarrow V^{\text{innen}} \approx V^{\text{innen}}(0) + \sum_{ij=1}^3 \frac{\partial V^{\text{innen}}}{\partial \eta_{ij}} \eta_{ij} + \frac{1}{2} \sum_{\substack{ij=1 \\ kl}}^3 \frac{\partial^2 V^{\text{innen}}}{\partial \eta_{ij} \partial \eta_{kl}} \eta_{ij} \eta_{kl}$$

$=: C_{ijkl}$ Elastizitätstensor

7.3 wave equation

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- circular / cylindrical membrane

$$\Delta u + u_{tt} = \frac{1}{c^2} u_{tt} \quad \text{with} \quad u(r=l, \varphi, t) = 0 \quad (\text{boundary condition})$$

protocol: Separation of (i) t and (r, φ) dependence

(ii) r and φ dependence

$$\Rightarrow u(r, \varphi, t) = R(r) \phi(\varphi) T(t) \quad , \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$$

$$(i) \quad T'' + c^2 k^2 T = 0 \quad \Rightarrow T(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$(ii) \quad \frac{d^2 \phi}{d\varphi^2} + \lambda^2 \phi = 0 \quad \Rightarrow \phi(\varphi) = C \cos(\lambda \varphi + \varphi_0) \quad \text{with } \lambda \in \mathbb{Z} \quad (2\pi \text{ periodicity})$$

$$(iii) \quad r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (k^2 r^2 - \lambda^2) R = 0 \quad \text{Bessel equation} \quad (x=kr)$$

$$\text{Ansatz: } R(k) = x^3 \sum_{s=0}^{\infty} a_s x^s$$

$$\Rightarrow R(r) = \sum_{m=0}^{\infty} D_m J_m(kr) \quad \text{with } J_m \text{ Bessel function}$$

$$\Rightarrow u(r, \varphi, t) = \sum_{k \in \mathbb{Z}} \sum_{m=0}^{\infty} \left[A_{km} \cos(\omega_{km} t) + B_{km} \sin(\omega_{km} t) \right] \cos(m\varphi + \varphi_0) J_m(k_{km} r)$$

$$\text{with } k_{km} = \frac{\mu_{km}}{l}, \quad \mu_{km}: \text{ } k\text{-th root of } J_m \quad (\text{boundary conditions})$$

- Lagrange density in 3D: $\mathcal{L} = \int d^3x \left[\frac{\rho}{2} \dot{\mathbf{q}}(k, t)^2 - \underbrace{V(\mathbf{q}, \nabla \mathbf{q})}_{\text{density of potential energy}} \right]$

$$V = V^{\text{internal}} + V^{\text{external}}$$

$$\uparrow \quad \quad \quad \uparrow \quad \quad \quad -f(\mathbf{r}, t) \cdot \mathbf{q}(\mathbf{r}, t)$$

$$\approx V^{\text{internal}}(0) + \sum_{j=1}^3 \frac{\partial V^{\text{internal}}}{\partial q_j} q_j + \frac{1}{2} \sum_{i,j,k,l} \frac{\partial^2 V^{\text{internal}}}{\partial q_i \partial q_j} q_i q_j + \dots$$

$$\mathbf{q} = \mathbf{r} \circ \mathbf{q} = \begin{pmatrix} q_x & q_y & q_z \\ \partial_x q_x & \partial_x q_y & \partial_x q_z \\ \partial_y q_x & \partial_y q_y & \partial_y q_z \\ \partial_z q_x & \partial_z q_y & \partial_z q_z \end{pmatrix}$$

C_{ijkl}
elastic tensor

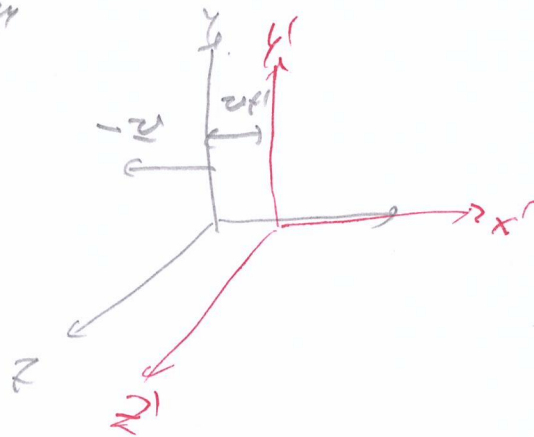
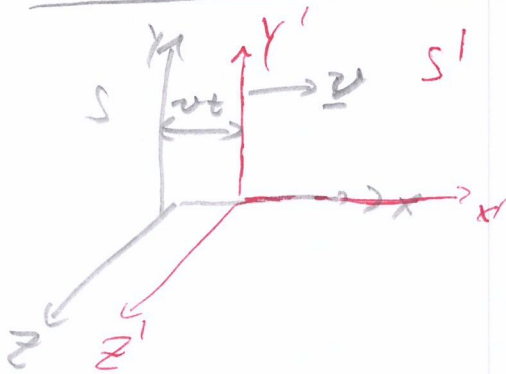
8. Spezielle Relativitätstheorie

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8.1 Postulate: (i) **Relativitätsprinzip:** In allen zueinander gleichförmig bewegten Bezugssystemen (Inertialsystemen) laufen physikalische Vorgänge bei gleichem Bedingungen gleich ab

(ii) **Konstanz der Lichtgeschwindigkeit:** In allen Inertialsystemen ist die Lichtgeschwindigkeit im Vakuum gleich groß (unabhängig vom Bewegungszustand der Quelle)

8.2 Lorentz-Transformation



Bewegung in x-Richtung:

$$x' = 0 \Leftrightarrow x = vt$$

$$x = 0 \Leftrightarrow x' = -vt'$$

$$y' = 0 \Leftrightarrow y = 0$$

$$z' = 0 \Leftrightarrow z = 0$$

$$\begin{cases} \text{Lorentz Transform: } x' = A(x - vt) \\ t' = Bt - Dx \end{cases} \Rightarrow B = A$$

Kugelwelle: $x^2 + y^2 + z^2 = c^2 t^2$ mit Lichtgeschwindigkeit c und $x'^2 + y'^2 + z'^2 = c^2 t'^2$ (Konstanz von c + Relativitätsprinzip)

$$\Rightarrow A^2(x^2 - 2xvt + v^2 t^2) + y^2 + z^2 = c^2(A^2 t^2 - 2ADtx + D^2 x^2)$$

Vergleiche die Terme: x^2 nach links, t^2 nach rechts

$$x^2: 1 = A^2 - C^2 D^2$$

$$xt: 0 = -2A^2 v + 2c^2 AD \Rightarrow D = \frac{v}{c^2} A =$$

$$t^2: C^2 = C^2 A^2 - A^2 v^2 \Rightarrow$$

$$\Rightarrow 1 = A^2 \left(1 - \frac{v^2}{c^2}\right) \Rightarrow A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

• $C \rightarrow \infty \Rightarrow A = 1, D = 0 \Rightarrow$ Galilei-Transformation (vgl. 1.3)

$$t' = t$$

$$x' = x - vt$$

mit $v = \beta c$ ($\beta = \frac{v}{c}$) und $\gamma = \frac{1}{\sqrt{1 - \beta^2}} : A = \gamma, D = \beta \gamma$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\underline{1(v)} \rightarrow \det(\underline{1(v)}) = 1$$

Inverse: $v \rightarrow -v$
 $\beta \rightarrow -\beta$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Für beliebige orientierte Geschwindigkeiten $\underline{v} : s^2 = c^2 t^2 - x^2 - y^2 - z^2 = \text{const}$

Satz 2.1.1

$$\begin{array}{ccc} S & \xrightarrow{\underline{A}(z)} & S' \xrightarrow{\underline{A}(w)} S'' \\ & \searrow & \nearrow \\ & \underline{A}(w) & \end{array}$$

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$$\Rightarrow \underline{A}(w') \underline{A}(v) = \underline{A}(w)$$

$$\begin{pmatrix} \gamma_{w'} & -\beta_{w'} \gamma_{w'} & 0 & 0 \\ \beta_{w'} \gamma_{w'} & \gamma_{w'} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_v & -\beta_v \gamma_v & 0 & 0 \\ -\beta_v \gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{w'} \gamma_v (1 + \beta_{w'} \beta_v) & -\gamma_{w'} \gamma_v (\beta_{w'} + \beta_v) & 0 & 0 \\ -\gamma_{w'} \gamma_v (\beta_{w'} + \beta_v) & \gamma_{w'} \gamma_v (1 + \beta_{w'} \beta_v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

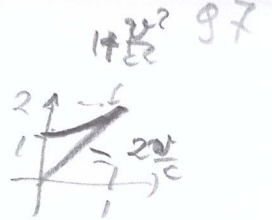
$$\stackrel{!}{=} \begin{pmatrix} \gamma_w & -\beta_w \gamma_w & 0 & 0 \\ -\beta_w \gamma_w & \gamma_w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \gamma_{w'} \gamma_v (1 + \beta_{w'} \beta_v) = \gamma_w, \quad \gamma_{w'} \gamma_v (\beta_{w'} + \beta_v) = \beta_w \gamma_w$$

$$\Rightarrow \frac{\beta_{w'} + \beta_v}{1 + \beta_{w'} \beta_v} = \beta_w \Leftrightarrow \frac{w' + v}{1 + w'v/c} = w$$

$$\left. \begin{array}{l} v < c \\ w' < c \end{array} \right\} \Rightarrow w < c$$

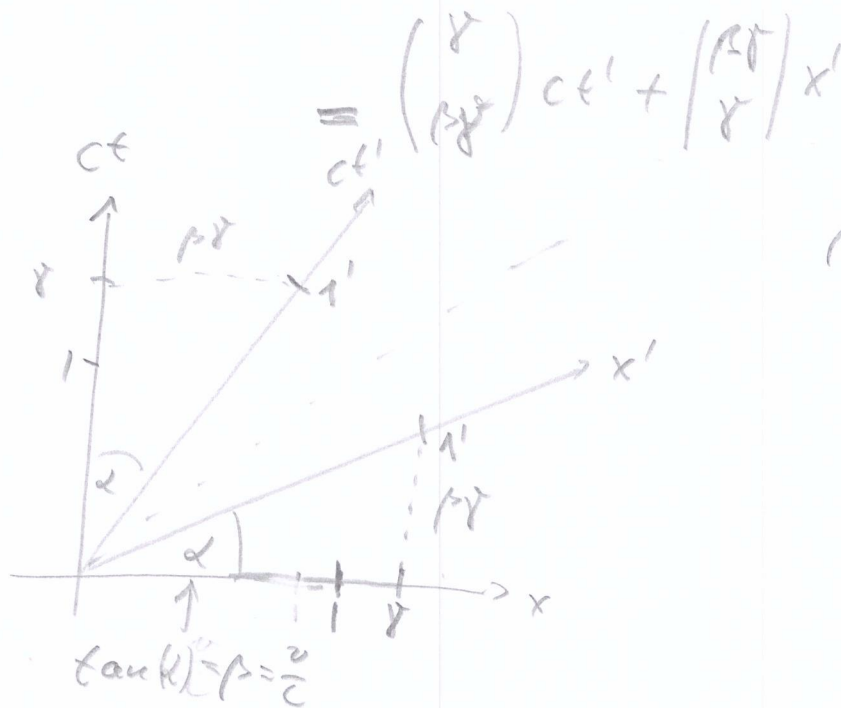
Satz: $w' < c \Rightarrow w < c$
Satz: $1 + \frac{v^2}{c^2}$



$$\left. \begin{array}{l} v = c \\ w' = c \end{array} \right\} \Rightarrow w = \frac{c + c}{1 + 1} = c$$

8.3 Minkowski-Diagramm

$$2D: \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \quad \left| \quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \right.$$

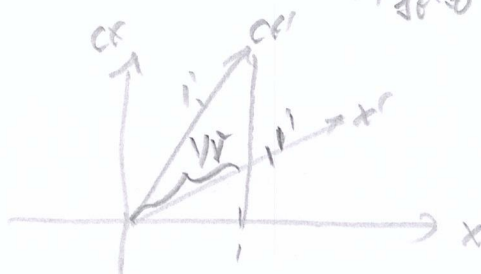
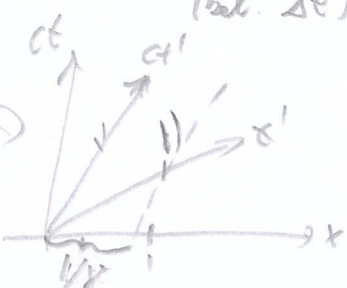


$$\beta\gamma = \frac{v}{c} \gamma < \gamma$$

(i) Längenkontraktion: $c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$, $c\Delta t = \gamma(c\Delta t' + \beta\Delta x')$
 $\Delta x' = \gamma(\Delta x - \beta c\Delta t)$, $\Delta x = \gamma(\Delta x' + \beta c\Delta t')$

L_0 in S' in Ruhe: $\Delta x' = L_0 \Rightarrow \text{in } S: \Delta x = \Delta x' \bigg|_{\Delta t=0} = \frac{\Delta x'}{\gamma} = \frac{L_0}{\gamma} < L_0$

L_0 in S in Ruhe: $\Delta x = L_0 \Rightarrow \text{in } S': \Delta x' = \Delta x \bigg|_{\Delta t=0} = \frac{\Delta x}{\gamma} = \frac{L_0}{\gamma} < L_0$

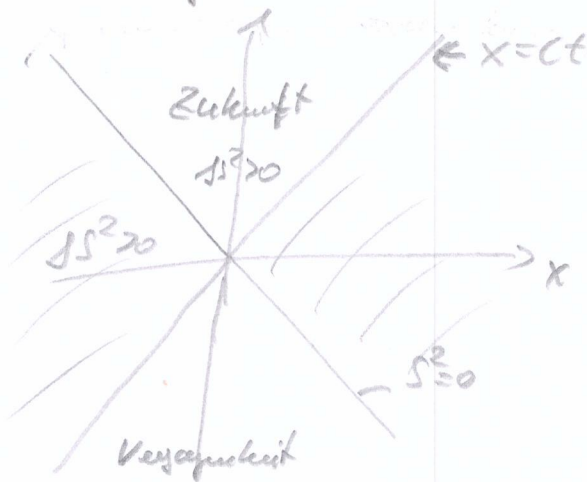


(ii) Zeitdilatation:

Uhr ruht in S' : $\Delta T_0 = \Delta t'$ in S : $\Delta T = \Delta t' / \sqrt{1 - v^2/c^2} = \gamma \Delta t' = \gamma \Delta T_0 > \Delta T_0$
 Bewegte Uhren gehen langsamer

Uhr ruht in S : $\Delta T_0 = \Delta t$ in S' : $\Delta T = \Delta t' / \sqrt{1 - v^2/c^2} = \gamma \Delta t = \gamma \Delta T_0 > \Delta T_0$

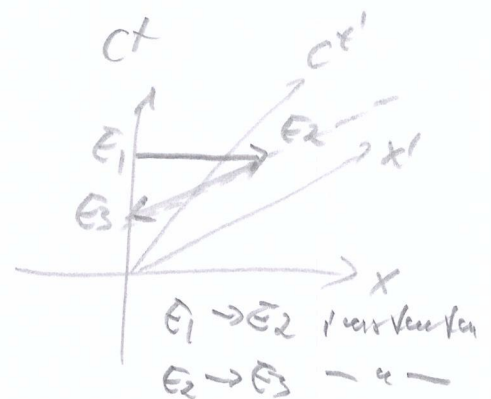
(iii) Lichtkegel ct



$$S^2 = c^2 dt^2 - dx^2$$

8.9 Summary:

- (i) Lebes dann notwendig von Mjoom
- (ii) Leiter Paradoxon
- (iii) Überlichtgeschwindigkeit ←
- (iv) GPS
- (v) Zwillingsparadoxon
- (vi) Ger-Formeln



relativistische Mechanik:

Formelssatz

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$$\underline{F} = \frac{d\underline{p}}{dt}$$

hier: $\epsilon' = \gamma t \Rightarrow \frac{d\epsilon'}{dt} = \gamma(v)$
mit γ benutzt

$$\frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \leq 1$$

mit $v' = \gamma(v)v$ ($\underline{p} = \underbrace{m \gamma(v)}_{=: m(v)} \underline{v}$)

mit Lorentzgeschwindigkeit:

$$\underline{u} = \frac{d\underline{x}}{dt} = \frac{d\underline{x}}{dt'} \frac{dt'}{dt}$$

$$\underline{u} = \gamma(v) \left(\frac{c}{\gamma} \right) \Rightarrow u^2 = c^2$$

$\Rightarrow$ Vorschau: $\underline{q} = \frac{d\underline{u}}{dt'} = \frac{d\underline{p}}{dt'^2} = \gamma \frac{d\underline{u}}{dt} = \gamma \frac{d}{dt} \gamma(v) \left(\frac{c}{\gamma} \right)$

$$\frac{d\underline{p}}{dt} = \frac{1}{c^2} \gamma^3 \underline{v} \cdot \underline{q} \left(\frac{d}{dt} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} + 2x \text{ Kettregel} \right)$$

$$\Rightarrow \underline{F} = m \gamma(v) \underline{q} + m \gamma^3(v) \frac{\underline{v} \cdot \underline{q}}{c^2} \underline{v}$$

$\underline{F}$ als Viererkräfte:

$$\Rightarrow 0 = \gamma (c \underline{F}^0 - \gamma \underline{F} \cdot \underline{v}) \Rightarrow \underline{F}^0 = \frac{\gamma}{c} \underline{F} \cdot \underline{v} = \frac{\gamma}{c} \underline{F} \cdot \frac{d\underline{x}}{dt} = \frac{\gamma}{c} \frac{\underline{F} \cdot d\underline{x}}{dt}$$

$$\frac{\gamma}{c} \frac{dW}{dt} = \frac{d}{dt'} \frac{W}{c}$$

mit $p^0 = \gamma m c$: $\frac{d}{dt'} W = \frac{d}{dt'} (c p^0) = \frac{d}{dt'} (\gamma(v) m c^2)$

$$\Rightarrow (p^0)^2 - \underline{p}^2 = \frac{\epsilon^2}{c^2} - \underline{p}^2 = m^2 c^2$$

$$\Rightarrow \epsilon = m c^2 \sqrt{1 + \frac{\underline{p}^2}{m^2 c^2}} = m c^2 \left(1 + \frac{1}{2} \frac{\underline{p}^2}{m^2 c^2} + \dots \right)$$

mit klassischer Term!