Fall a) $T = \frac{1}{2} m \dot{q}^2$

$$V(q) = mg\,z(q) = mg\ell(1-\cos\varphi(q)) = mg\ell(1-\cos\frac{q}{\ell})$$

$$\left.\begin{array}{l} \Rightarrow \left(\frac{\partial V}{\partial q}\right)_0 = mg\frac{\ell}{\ell}\sin\frac{q}{\ell}\Big|_0 = 0 \\ \left(\frac{\partial^2 V}{\partial q^2}\right)_0 = m\frac{g}{\ell}\cos\frac{q}{\ell}\Big|_0 = m\frac{g}{\ell} \end{array}\right\} \Rightarrow V(q) \approx \frac{1}{2}\left(\frac{\partial^2 V}{\partial q^2}\right)_0 q^2 = \frac{1}{2} m \frac{g}{\ell} q^2$$

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} \overset{L=T-V}{=} \frac{d}{dt}\left(m\dot{q}\right) + m\frac{g}{\ell}q$$

$$= m\ddot{q} + m\frac{g}{\ell}q$$

$$\Leftrightarrow \quad 0 = \ddot{q} + \frac{g}{\ell}q \quad \text{oder} \quad 0 = \ddot{q} + \omega^2 q \quad \text{mit } \omega^2 = \frac{g}{\ell}$$

$\boxed{\text{Lösg: } q(t) = Ae^{i\omega t} + Be^{-i\omega t}}$

Fall b): $T = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2)$

$$V(q_1, q_2) = mg\ell(1-\cos\frac{q_1}{\ell}) + mg\ell(1-\cos\frac{q_2}{\ell}) + \underbrace{\frac{k}{2}(q_1-q_2)^2}_{\text{Hooke'sches Gesetz}}$$

$$\left(\frac{\partial^2 V}{\partial q_1^2}\right)_0 = \left(\frac{\partial^2 V}{\partial q_2^2}\right)_0 = m\frac{g}{\ell} + k$$

$$\left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 = mg\underbrace{\frac{\partial}{\partial q_1}\left(\sin\frac{q_2}{\ell}\right)}_{=0} - k^2\frac{\partial}{\partial q_1}(q_1 - q_2) = -k = \left(\frac{\partial^2 V}{\partial q_2 \partial q_1}\right)_0$$

$$\Rightarrow V(q_1, q_2) \approx \frac{1}{2} m\frac{g}{\ell}(q_1^2 + q_2^2) + \frac{1}{2}k(q_1 - q_2)^2$$

in Matrixschreibweise:

$$\underline{\underline{T}} \equiv \{T_{jk}\}_{j,k=1,2} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \{V_{jk}\}_{j,k=1,2} = \begin{pmatrix} m\frac{g}{\ell} + k & -k \\ -k & m\frac{g}{\ell} + k \end{pmatrix} \equiv \underline{\underline{V}}$$

Bewegungsgleichungen: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2$

$$\left.\begin{array}{l} m\ddot{q}_1 + m\frac{g}{\ell}q_1 + k(q_1 - q_2) = 0 \\ m\ddot{q}_2 + m\frac{g}{\ell}q_2 - k(q_1 - q_2) = 0 \end{array}\right\} \text{lineare gekoppelte DGLs}$$

Lösungsansatz: $q_j = A_j e^{i\omega t} \Rightarrow \ddot{q}_j = -\omega^2 A_j e^{i\omega t}$

$\Rightarrow$ Eigenwertgleichung: $\underbrace{\begin{pmatrix} m\frac{g}{l}+k-m\omega^2 & -k \\ -k & m\frac{g}{l}+k-m\omega^2 \end{pmatrix}}_{=\underline{\underline{V}}-\omega^2\underline{\underline{T}}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$

$\Rightarrow$ charakteristische Gleichung:

$$0 = m^2 \begin{vmatrix} \frac{g}{l}+\frac{k}{m}-\omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l}+\frac{k}{m}-\omega^2 \end{vmatrix}$$

$$= \omega^4 - 2\omega^2\left(\frac{g}{l}+\frac{k}{m}\right) + \left(\frac{g}{l}+\frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2$$

$$\Rightarrow \omega_{1,2}^2 = \left(\frac{g}{l}+\frac{k}{m}\right) \pm \frac{k}{m} = \begin{cases} \frac{g}{l} \\ \frac{g}{l}+2\frac{k}{m} \end{cases}$$

Eigenfrequenzen: $\omega_1 = \sqrt{\frac{g}{l}}$: ungestörte Eigenfrequenz

$\omega_2 = \sqrt{\frac{g}{l}+\frac{2k}{m}} =: \sqrt{\omega_0^2 + 2\tilde{\omega}^2}$

Eigenvektoren:

zu ω_1: $kA_1^{(1)} - kA_2^{(1)} = 0 \Rightarrow \begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \underset{\uparrow}{=} \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Normierung $m \sum_k |A_k|^2 = 1 = \sum_{jk} A_j^{(i)} T_{jk} A_k^{(i)}$

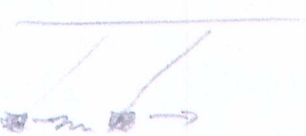
zu ω_2: $-kA_1^{(1)} - kA_2^{(2)} = 0 \Rightarrow \begin{pmatrix} A_1^{(2)} \\ A_2^{(2)} \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} \overset{!}{=} \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

allgemeine Lsg: $\begin{pmatrix} A_1^{(1)} & A_2^{(1)} \\ A_1^{(2)} & A_2^{(2)} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} q_1+q_2 \\ q_1-q_2 \end{pmatrix}$ ← Schwerpunktkoordinate ← Relativkoordinate

$= \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \Rightarrow$ Umkehrung auch möglich

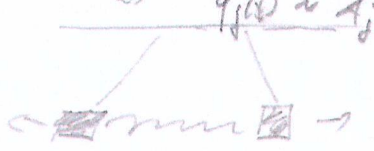
$q_j(t) \sim A_j^{(1)} Q_1 + A_j^{(2)} Q_2$

Normalschwingungen:



$\omega_1 = \sqrt{\frac{g}{l}}$

gleichphasig



$\omega_2 = \sqrt{\frac{g}{l}+2\frac{k}{m}}$

gegenphasig

$\Rightarrow$ Überlagern: Schwebung

Normalschwingungen ganz allgemein:

$$L = T - V = \frac{1}{2}\sum_{jk=1}^{f}\left(T_{jk}\,\dot q_j \dot q_k - V_{jk}\, q_j q_k\right)$$

$$\hookrightarrow \frac{\partial L}{\partial \dot q_l} = \frac{1}{2}\sum_{jk=1}^{f} T_{jk} \underbrace{\frac{\partial}{\partial \dot q_l}(\dot q_j \dot q_k)}_{\substack{= \frac{\partial \dot q_j}{\partial \dot q_l}\dot q_k + \dot q_j \frac{\partial \dot q_k}{\partial \dot q_l} \\ = \delta_{jl}\dot q_k + \dot q_j \delta_{kl}}} = \frac{1}{2}\sum_k T_{lk}\dot q_k + \frac{1}{2}\underbrace{\sum_j T_{jl}\dot q_j}_{=\sum_k T_{lk}\dot q_k} = \sum_k T_{lk}\dot q_k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_l}\right) = \sum_k T_{lk}\ddot q_k$$

$$\frac{\partial V}{\partial q_l} = \frac{1}{2}\sum_{jk=1}^{f} T_{jk}\frac{\partial}{\partial q_l}(q_j q_k) = \ldots = \sum_k V_{lk} q_k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_l}\right) - \frac{\partial L}{\partial q_l} = 0 \Leftrightarrow \sum_k \left(T_{lk}\ddot q_k + V_{lk} q_k\right) = 0\,,\quad l = 1,\ldots,f$$

lineare DGLs!

Ansatz: $q_k(t) = A_k e^{i\omega t}$

$$\Rightarrow \sum_{k=1}^{f}\left(V_{lk} - \omega^2 T_{lk}\right) A_k = 0$$

- Eigenwertgleichung für ω^2! ω_α^2, $\alpha = 1,\ldots,f$
- homogenes, lineares GLS für $A_k \to A_k^{(\alpha)}$

Nichttriviale Lsg genau dann, wenn $\det\left(V_{lk} - \omega^2 T_{lk}\right) = 0$

Charakt. Gleichung für ω^2

allg. Lsg: $q_k(t) = \mathrm{Re}\left\{\sum_{\alpha=1}^{f} C_\alpha A_k^{(\alpha)} e^{i\omega_\alpha t}\right\}$

Bestimmt durch Anfangsbed. $q_k(0), \dot q_k(0)$

V, T pos definit $\Rightarrow \omega^2 > 0$

$$\sum_{k=1}^{f}\left(V_{lk} - \omega^2 T_{lk}\right) A_k = 0 \quad \Big| \cdot \sum_{l=1}^{f} A_l^*$$

$$\sum_{lk=1}^{f} V_{lk} A_l^* A_k - \omega^2 \sum_{lk=1}^{f} T_{lk} A_l^* A_k = 0 \Rightarrow \omega^2 = \frac{\sum_{lk=1}^{f} V_{lk} A_l^* A_k}{\sum_{lk=1}^{f} T_{lk} A_l^* A_k}$$

Normalkoordinaten:

Idee: Transformiere auf neue Koordinaten Q_α, sodass die Bewegungsgleichungen entkoppeln $\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$, $\alpha = 1, \dots, f$

$\Rightarrow$ Hauptachsentrafo der symmetrischen Matrizen V_{lk}, T_{lk}

$\Rightarrow$ Diagonalisierung durch Trafo mit (reell gewählten) Eigenvektoren $A_k^{(\alpha)}$:

$$q_k(t) = \sum_{\alpha=1}^{f} A_k^{(\alpha)} Q_\alpha(t) \Rightarrow \underline{q} = \underline{\underline{A}}\,\underline{Q}$$

Check Diagonalformieren V_{lk}, T_{lk} (gleichzeitig diagonalisierbar)

$$\text{I}: \sum_{k=1}^{f} (V_{lk} - \omega_\alpha^2 T_{lk}) A_k^{(\alpha)} = 0 \quad \Big| \cdot \sum_{l=1}^{f} A_l^{(\beta)}$$

$$\text{II}: \sum_{l=1}^{f} (V_{kl} - \omega_\beta^2 T_{kl}) A_l^{(\beta)} = 0 \quad \Big| \cdot \sum_{k=1}^{f} A_k^{(\alpha)}$$

$$\Rightarrow \text{I}-\text{II}: \sum_{kl=1}^{f} \underbrace{(V_{lk} - V_{kl})}_{=0} A_l^{(\beta)} A_k^{(\alpha)} - (\omega_\alpha^2 T_{lk} - \omega_\beta^2 T_{kl}) A_l^{(\beta)} A_k^{(\alpha)} = 0$$

$$\Rightarrow \underbrace{(\omega_\alpha^2 - \omega_\beta^2)}_{\neq 0 \text{ für } \alpha \neq \beta} \underbrace{\sum_{kl=1}^{f} A_l^{(\beta)} T_{lk} A_k^{(\alpha)}}_{T^{(\alpha\beta)} \overset{!}{=} 0 \text{ für } \alpha \neq \beta} = 0$$

(transformierte kinetische Energie)

$$\Rightarrow \sum_{kl=1}^{f} A_l^{(\beta)} T_{lk} A_k^{(\alpha)} = \delta_{\beta\alpha}$$

(geeignete Normierung der Eigenvektoren A_k für $\alpha = \beta$)

$$\Rightarrow \sum_{kl=1}^{f} A_l^{(\beta)} V_{lk} A_k^{(\alpha)} = \omega_\alpha^2 \sum_{kl=1}^{f} A_l^{(\beta)} T_{lk} A_k^{(\alpha)} = \omega_\alpha^2 \delta_{\beta\alpha}$$

$\Rightarrow$ T_{lk} & V_{lk} gleichzeitig diagonalisiert

$$\Rightarrow L = \frac{1}{2} \sum_{jk=1}^{f} (T_{jk} \dot{q}_j \dot{q}_k - V_{jk} q_j q_k)$$

$$= \frac{1}{2} \sum_{\alpha\beta=1}^{f} \Big(\underbrace{\sum_{jk=1}^{f} A_j^{(\beta)} T_{jk} A_k^{(\alpha)}}_{\delta_{\beta\alpha}} \dot{Q}_\alpha \dot{Q}_\beta - \underbrace{\sum_{jk=1}^{f} A_j^{(\beta)} V_{jk} A_k^{(\alpha)}}_{= \omega_\alpha^2 \delta_{\beta\alpha}} Q_\alpha Q_\beta \Big)$$

$$= \frac{1}{2} \sum_{\alpha=1}^{f} (\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2) \Rightarrow \text{Entkoppelt: } \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_\alpha} - \frac{\partial L}{\partial Q_\alpha} = 0 = \ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0$$