2.7.1 Impuls- & Drehimpulserhaltung

Idee: Lagrange-Formalismus: $L = T - V = \frac{1}{2} m_1 \dot{\underline{r}}_1^2 + \frac{1}{2} m_2 \dot{\underline{r}}_2^2 - V(|\underline{r}_1 - \underline{r}_2|)$

Verallgemeinerte Koordinaten:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underline{R} = \frac{1}{M}(m_1 \underline{r}_1 + m_2 \underline{r}_2) \qquad \text{Schwerpunktskoordinate}$$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \end{pmatrix} = \underline{r} = \underline{r}_1 - \underline{r}_2 \qquad \text{Relativkoordinate}$$

$$\hookrightarrow \underline{r}_1 = \underline{R} + \frac{m_2}{M}\underline{r}, \quad \underline{r}_2 = \underline{R} - \frac{m_1}{M}\underline{r}$$

$$\dot{\underline{r}}_1 = \dot{\underline{R}} + \frac{m_2}{M}\dot{\underline{r}}, \quad \dot{\underline{r}}_2 = \dot{\underline{R}} - \frac{m_1}{M}\dot{\underline{r}}$$

$$\Rightarrow L \underset{\uparrow}{=} \frac{M}{2}\dot{\underline{R}}^2 + \frac{1}{2} m \dot{\underline{r}}^2 - V(r), \quad r = |\underline{r}| \text{ Abstand}, \quad m = \frac{m_1 m_2}{m_1 + m_2} \text{ reduzierte Masse}$$

nachrechnen!

- $\frac{\partial L}{\partial R_k} = 0 \Rightarrow \underline{R}$ ist zyklische Variable $\Rightarrow \frac{\partial L}{\partial \dot{R}_k} = M\dot{R}_k = P_k = \text{const}$

  $\Rightarrow$ zugehöriger generalisierter Impuls = const. $\quad \Rightarrow \underline{R} = \frac{1}{M}\underline{P}t + \underline{R}_0$

- Schwerpunktsystem als Inertialsystem! OBdA $\underline{R} = \dot{\underline{R}} = 0$

$$\Rightarrow L = \frac{1}{2} m \dot{\underline{r}}^2 - V(r), \quad \underline{r}_1 = \frac{m_2}{M}\underline{r}, \quad \underline{r}_2 = -\frac{m_1}{M}\underline{r}$$
$$\dot{\underline{r}}_1 = \frac{m_2}{M}\dot{\underline{r}}, \quad \dot{\underline{r}}_2 = -\frac{m_1}{M}\dot{\underline{r}}$$

Drehimpuls: $\underline{\ell} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2$

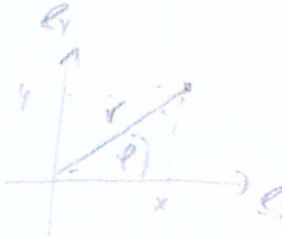
$$= m_1 \frac{m_2}{M}\underline{r} \times \frac{m_2}{M}\dot{\underline{r}} + m_2\left(-\frac{m_1}{M}\right)\underline{r} \times \left(-\frac{m_1}{M}\right)\dot{\underline{r}}$$

$$= \underbrace{\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right)}_{\frac{(m_1+m_2) m_1 m_2}{M^2} = \frac{m_1 m_2}{M} = m} \underline{r} \times \dot{\underline{r}} \underset{\uparrow}{=} m\, \underline{r} \times \dot{\underline{r}} = \text{const}$$

nachrechnen

$\Rightarrow \underline{\ell} \cdot \underline{r} = 0, \ \underline{\ell} \cdot \dot{\underline{r}} = 0 \Rightarrow \underline{r}$ und $\dot{\underline{r}}$ in Ebene senkrecht zu $\underline{\ell}$ (im Schwerpunktsystem)

$\Rightarrow$ 2D Bewegung! $\Rightarrow$ Polarkoordinaten
(oBdA $\underline{l} \parallel \underline{e}_z$)



$x = r\cos\varphi \Rightarrow \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$

$y = r\sin\varphi \Rightarrow \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$

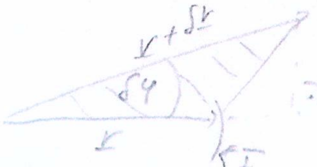
$$\Rightarrow \dot{\underline{r}}^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2(\cos^2\varphi + \sin^2\varphi) + r^2\dot{\varphi}^2(\sin^2\varphi + \cos^2\varphi) = \dot{r}^2 + r^2\dot{\varphi}^2$$

$$\Rightarrow L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

- $\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi$ ist zyklisch $\Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = \underset{\uparrow \atop m(x\dot{y} - y\dot{x}) = mr^2\dot{\varphi}}{l_z} \overset{l_x = l_y = 0 \atop \downarrow}{=} l = \text{const}$

- **2. Keplersches Gesetz**: Flächensatz (geometrische Interpretation von $m r^2 \dot{\varphi} = l = \text{const}$)

Der Radiusvektor $\underline{r}$ überstreicht in gleichen Zeiten gleiche Flächen.
(Flächengeschwindigkeit $\frac{dF}{dt} = \text{const}$)



$\Rightarrow \Big\} \; |\underline{r} + \delta\underline{r}| \sin\delta\varphi$

$$\delta F = \frac{1}{2}|\underline{r}|\,|\underline{r} + \delta\underline{r}| \sin\delta\varphi \underset{\uparrow \atop {\delta r, \delta\varphi \to 0 \atop \sin\delta\varphi \approx \delta\varphi}}{\approx} \frac{1}{2}|\underline{r}|\,|\underline{r}|\,\delta\varphi$$

$$\Rightarrow \frac{dF}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2} r^2 \dot{\varphi} \underset{\uparrow \atop m r^2 \dot{\varphi} = l}{=} \frac{l}{2m} = \text{const!}$$