2.7.2 Energieerhaltung und Bahngleichung

Lagrange-Gleichung 2. Art für r: $\dfrac{d}{dt}\left(\underbrace{\dfrac{\partial L}{\partial \dot r}}_{m\dot r}\right) - \underbrace{\dfrac{\partial L}{\partial r}}_{mr\dot\varphi^2 - V'(r)} = 0$

$$\Rightarrow \quad m\ddot r - \underbrace{m r \dot\varphi^2}_{\text{Zentrifugalkraft}} + V'(r) = 0 \qquad \dot\varphi = \frac{l}{mr^2}$$

$$\Rightarrow \quad m\ddot r - \frac{l^2}{mr^3} + V'(r) = 0 \qquad \Big| \cdot \dot r$$

$\uparrow$ $\dot\varphi = \dfrac{l}{mr^2}$

$$\underbrace{m\dot r \ddot r}_{\frac{d}{dt}\left(\frac{m}{2}\dot r^2\right)} + \frac{l^2}{m}\underbrace{\left(-\frac{\dot r}{r^3}\right)}_{\frac{d}{dt}\frac{1}{2r^2}} + \underbrace{\dot r V'(r)}_{\frac{d}{dt}V(r)}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{m}{2}\dot r^2 + \frac{l^2}{2mr^2} + V(r)\right) = 0 \qquad \int dt$$

$$\underbrace{\frac{m}{2}\dot r^2 + \frac{l^2}{2mr^2}}_{\frac{m}{2}\left(\dot r^2 + \frac{m r^4 \dot\varphi^2}{m r^2}\right) = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2\right) = T} + V(r) = E = \text{const} \qquad E = T + V$$

Alternativ: $E = \dfrac{m}{2}\dot r^2 + \tilde V(r)$ mit effektivem Radialpotential $\tilde V(r) = V(r) + \dfrac{l^2}{2mr^2}$

$$\Rightarrow \quad \dot r = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - \tilde V(r)\right)}$$

Trennung der Variablen:

$$\int_{t_0}^{t} dt' = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left(E - \tilde V(r')\right)}} \quad \Rightarrow \quad t = t(r) \Rightarrow \text{Umkehrung: } r = r(t)$$

$$\Rightarrow \quad \varphi(t) \text{ aus } \dot\varphi = \frac{d\varphi}{dt} = \frac{l}{m r^2(t)}: \quad \int_{\varphi_0}^{\varphi} d\varphi' = \int_{t_0}^{t} dt' \, \frac{l}{m r^2(t')} \quad \Rightarrow \quad \varphi = \varphi(t)$$

Bahngleichung: $r = r(\varphi)$, $\varphi = \varphi(r)$

$$\frac{dr}{d\varphi} = \frac{\dot r}{\dot\varphi} = \frac{mr^2}{\ell}\sqrt{\frac{2}{m}\left(E-\tilde V(r)\right)}$$

$$\int d\varphi = \int \frac{dr}{r^2\sqrt{\frac{2m}{\ell^2}\left(E-\tilde V(r)\right)}} = \varphi - \varphi_0$$

2.7.3 Planetenbewegung & Keplersche Gesetze

Nun: Gravitationspotential: $V(r) = -\frac{\gamma m_1 m_2}{r}$

$$\Rightarrow \tilde V(r) = -\frac{K}{r} + \frac{\ell^2}{2mr^2}\,, \quad K = \gamma m_1 m_2 > 0$$

$r \to 0$: $\tilde V(r) \sim \frac{\ell^2}{2mr^2} \to \infty$

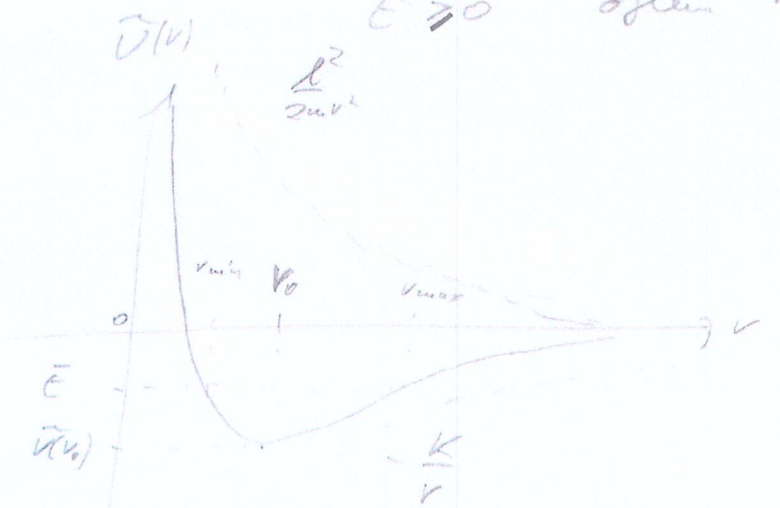
$r \to \infty$: $\tilde V(r) \sim -\frac{K}{r} \to 0$

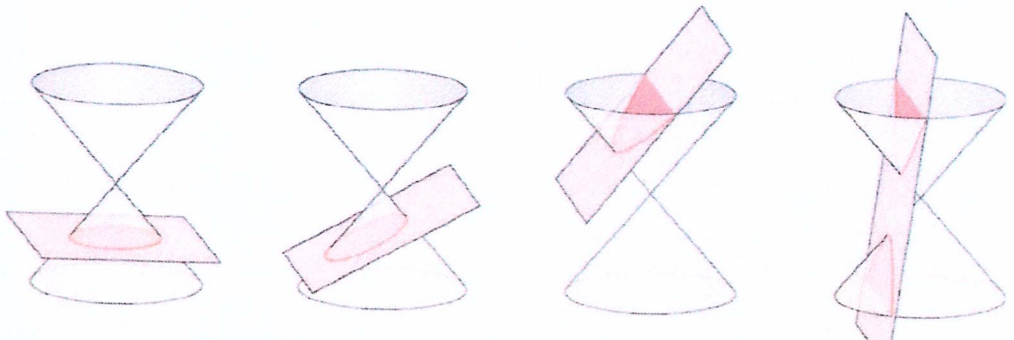
Minimum: $\frac{d\tilde V}{dr} = \frac{K}{r^2} - \frac{2\ell^2}{2mr^3} \overset{!}{=} 0 \Rightarrow r_0 = \frac{\ell^2}{mK} \Rightarrow \tilde V(r_0) = -\frac{mK^2}{2\ell^2}$

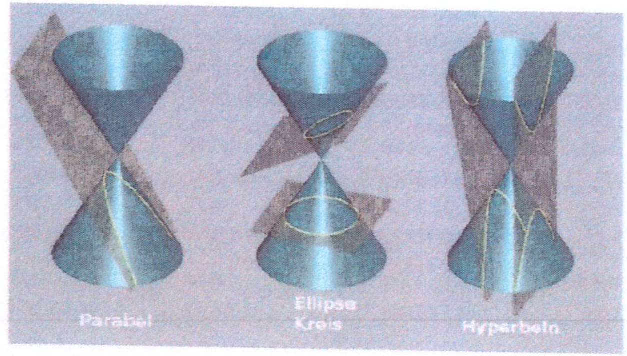
Wegen $\frac{m}{2}\dot r^2 = E - \tilde V(r)$ ist eine Bewegung nur für $\tilde V(r) < E$ möglich

$\Rightarrow -\frac{mK^2}{2\ell^2} < E < 0$ geschlossene Bahn (Ellipse)

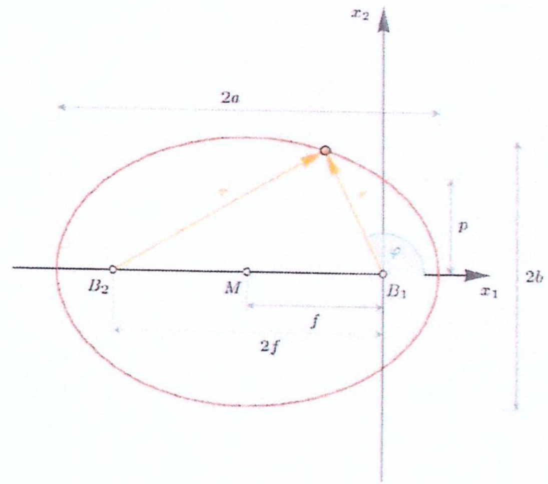
$E \geq 0$ offene Bahn $\underbrace{\left(\underset{E>0}{\text{Hyperbel}}, \underset{E=0}{\text{Parabel}}\right)}_{\text{Kegelschnitte!}}$



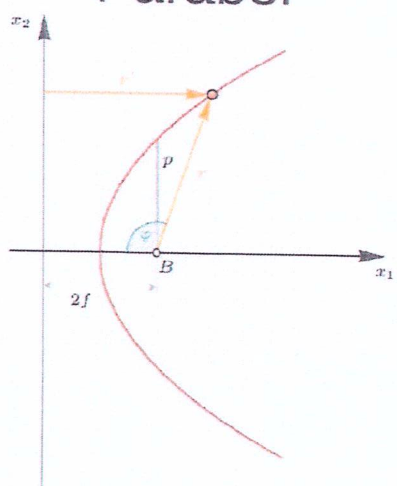




Ellipse



Parabel



Hyperbel