Für $-\frac{mk^2}{2\ell^2} < E < 0$ sind die Wendepunkte durch

$$\tilde{V}(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} = E \quad \text{bestimmt } (E<0)$$

$$\Rightarrow r_{min/max} = \frac{1}{2|E|}\left(k \pm \sqrt{k^2 - \frac{2\ell^2|E|}{m}}\right)$$

$$E r^2 + kr - \frac{\ell^2}{2m} = 0$$

$$r^2 - \frac{k}{|E|} r + \frac{\ell^2}{2m|E|} = 0$$

$$\hookrightarrow r_{min/max} = \frac{1}{2}\frac{k}{|E|} \mp \sqrt{\frac{k^2}{|E|^2} - 4\frac{\ell^2}{2m}}$$

$$= \frac{1}{2}\frac{1}{|E|}\left(k \mp \sqrt{k^2 - \frac{2\ell^2|E|}{m}}\right)$$

$E>0$: nur 1 Wendepunkt: $r_{min} = \frac{1}{2E}\left(\sqrt{k^2 - \frac{2\ell^2 E}{m}} - k\right)$

Bahngleichung: $$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}(E - \tilde{V}(r'))}}$$

$$= \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}E + \frac{2mk}{\ell^2 r'} - \frac{1}{r'^2}}}$$

$$= \dots = \int_{\vartheta_0}^{\vartheta} d\vartheta' \quad \text{mit } \cos\vartheta = \frac{1}{\sqrt{D}}\left(\frac{1}{r} - \frac{mk}{\ell^2}\right)$$

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Lösen des Integrals:

$$\varphi-\varphi_0=\int_{r_0}^{r}\frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}\left(E-\tilde V(r')\right)}}=\int_{r_0}^{r}\frac{dr'}{r'^2\sqrt{\frac{2m}{\ell^2}\left(E+\frac{k}{r'}-\frac{\ell^2}{2mr'^2}\right)}}$$

durch (i) quadratische Ergänzung und (ii) Substitution

(i) $\frac{2m}{\ell^2}\left(E-\tilde V(r')\right)=\underbrace{\frac{2mE}{\ell^2}}+\frac{2mk}{\ell^2 r'}-\frac{1}{r'^2}=\underbrace{-\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2}_{=-\frac{1}{r'^2}+\frac{2mk}{\ell^2 r'}-\frac{m^2k^2}{\ell^4}}+\underbrace{\frac{m^2k^2}{\ell^4}+\frac{2mE}{\ell^2}}_{=\frac{2m}{\ell^2}\left(\frac{mk^2}{2\ell^2}+E\right)}$

$$=D-\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2$$

$$=D\left[1-\frac{1}{D}\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2\right]\quad\text{mit } D=\frac{2m}{\ell^2}\left(\frac{mk^2}{2\ell^2}+E\right)>0$$

(ii) Substitution: $\cos\vartheta'=\frac{1}{\sqrt{D}}\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)\Rightarrow-\sin\vartheta'\,d\vartheta'=-\frac{dr'}{\sqrt{D}\,r'^2}$

$\frac{d}{dr'}\cos\vartheta'(r')=\frac{d}{dr'}(\dots)$

$$\Rightarrow\varphi-\varphi_0=\int_{r_0}^{r}\frac{dr'}{r'^2}\frac{1}{\sqrt{D}}\frac{1}{\sqrt{1-\frac{1}{D}\left(\frac{1}{r'}-\frac{mk}{\ell^2}\right)^2}}=\int_{\vartheta_0}^{\vartheta}d\vartheta'\sin\vartheta'\overbrace{\frac{1}{\sqrt{1-\cos^2\vartheta'}}}^{\frac{1}{\sin\vartheta'}}=\int_{\vartheta_0}^{\vartheta}d\vartheta'$$

$$=\vartheta-\vartheta_0=\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r}-\frac{mk}{\ell^2}\right)\right]-\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r_0}-\frac{mk}{\ell^2}\right)\right]$$

r_0,φ_0 Integrationskonstanten!

wähle: $\varphi_0=\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r_0}-\frac{mk}{\ell^2}\right)\right]\Rightarrow\varphi(r)=\arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r}-\frac{mk}{\ell^2}\right)\right]$

$$\Rightarrow\frac{1}{r(\varphi)}=\frac{mk}{\ell^2}+\sqrt{D}\cos\varphi=\frac{mk}{\ell^2}\left(1+\varepsilon\cos\varphi\right)$$

$$\varepsilon:=\sqrt{D}\,\frac{\ell^2}{mk}=\sqrt{1+\frac{2E\ell^2}{mk^2}}\geq 0$$

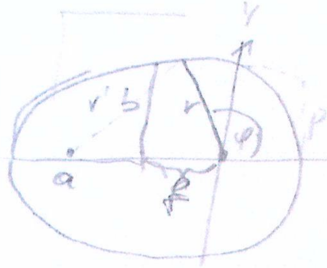
$$\Leftrightarrow r(\varphi)=\frac{\frac{\ell^2}{mk}}{1+\varepsilon\cos\varphi}\quad\begin{cases}\varepsilon=0 & r(\varphi)=\frac{\ell^2}{mk}=\text{const}\Rightarrow\text{Kreis}\\ 0<\varepsilon<1 & \text{Ellipse}\end{cases}$$

Kegelschnitte:

$\varepsilon > 1 \;\hat{=}\; E > 0$ Hyperbel — offene Bahn — $r' - r = 2a$

$\varepsilon = 1 \;\hat{=}\; E = 0$ Parabel — " —

$0 < \varepsilon < 1 \;\hat{=}\; \underbrace{-\frac{mk^2}{2l^2}}_{V_0(r_0)\ \text{Minimum}} < E < 0$ Ellipse — geschlossene Bahn — $r' + r = 2a$



$\varepsilon < 1$: $\cos\varphi = \frac{x}{r}$, $r = \sqrt{x^2+y^2}$

$\Rightarrow \boxed{\frac{(x+e)^2}{a^2} + \frac{y^2}{b^2} = 1}$, $f = \sqrt{a^2-b^2}$ lineare Exzentrizität — Exzentrizität

Ellipsengleichung

$r' + r = 2a = const$

$\varepsilon = \frac{f}{a}$: numerische Exzentrizität

Hauptachsen: $a = \frac{l^2}{mk(1-\varepsilon^2)} = \frac{k}{2|E|}$

$(1-\varepsilon^2)\,a = \frac{l^2}{mk}$

$\Rightarrow r(\varphi) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi}$

$b = \frac{l^2}{mk\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2m|E|}}$

$\Rightarrow \frac{b^2}{a} = \frac{l^2}{mk}$

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alternative Ellipsengleichung:

$$\frac{1}{r} = \frac{mk}{l^2} + \sqrt{D}\,\frac{x}{r} = \frac{mk}{l^2}\left(1 + \frac{\varepsilon x}{r}\right)$$

$$\Leftrightarrow \quad 1 = \frac{mk}{l^2}\sqrt{x^2+y^2} + \frac{mk}{l^2}\varepsilon x$$

$$\Leftrightarrow \quad \sqrt{x^2+y^2} = \frac{l^2}{mk} - \varepsilon x$$

$$\Leftrightarrow \quad x^2+y^2 = \left(\frac{l^2}{mk}\right)^2 - \frac{l^2 2\varepsilon x}{mk} + \varepsilon^2 x^2$$

$$\Leftrightarrow \quad y^2 = \left(\frac{l^2}{mk}\right)^2 - \frac{l^2 2\varepsilon}{mk}x + (\varepsilon^2-1)x^2$$

3. Keplersches Gesetz

$T^2 \sim a^3$ (T: Umlaufszeit, a: große Halbachse)

Ellipsenfläche $F = \pi a b$

mit $\frac{dF}{dt} = \frac{l}{2m} = \text{const}$ folgt: $F = \pi a b \int_0^T \frac{dF}{dt} dt = \frac{l}{2m} T$

$$\Rightarrow T = \frac{2m\pi ab}{l} \underset{\uparrow}{=} \frac{2m\pi a}{l} \frac{l}{\sqrt{mk}} \sqrt{a} = 2\pi \sqrt{\frac{m}{k}} a^{3/2}$$

$$\frac{b^2}{a} = \frac{l^2}{mk} \Rightarrow b = \frac{l}{\sqrt{mk}} \sqrt{a}$$

$$\Rightarrow T^2 \sim a^3$$

mit $m = \frac{m_1 m_2}{m_1 + m_2}$, $k = \gamma m_1 m_2$ folgt: $\frac{m}{k} = \frac{1}{\gamma(m_1 + m_2)}$

Für m_1 (Planet) $\ll m_2$ (Sonne): $\frac{m}{k} \approx \frac{1}{\gamma m_2} \Rightarrow T \approx 2\pi \frac{a^{3/2}}{\sqrt{\gamma m_2}}$