## 4. Wellen

Verwende **Lorentz-Eichung** $\nabla\cdot A + \epsilon_0\mu_0 \frac{\partial}{\partial t}\phi = 0$

(i) $-\nabla\cdot E = \nabla\left(\nabla\phi + \frac{\partial}{\partial t}A\right) = -\frac{1}{\epsilon_0}\rho$

$$\Rightarrow \Delta\phi + \frac{\partial}{\partial t}\nabla\cdot A = -\frac{1}{\epsilon_0}\rho$$

Lorentz-Eichung

$$\Downarrow$$

$$\Rightarrow \Delta\phi - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\phi = -\frac{1}{\epsilon_0}\rho$$

(ii) $\frac{1}{\mu_0}\nabla\times B - \epsilon_0\frac{\partial}{\partial t}E = j$

$$\Rightarrow \underbrace{\nabla\times(\nabla\times A)}_{\nabla(\nabla\cdot A)-\Delta A} + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial}{\partial t}A\right) = \mu_0 j$$

$$\Rightarrow \Delta A - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}A - \nabla\underbrace{\left(\nabla\cdot A + \epsilon_0\mu_0\frac{\partial}{\partial t}\phi\right)}_{=0\ \text{(Lorentz-Eichung)}} = -\mu_0 j$$

Zusammengefasst mit $\Box := \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ **d'Alembert-Operator**

$\Box\phi = -\frac{1}{\epsilon_0}\rho$ **inhomogene Wellengleichung**

$\Box A = -\mu_0 j$ und $c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2{,}994\times 10^8\,\frac{m}{s}$

**Lichtgeschwindigkeit**

im Vakuum: $\rho = 0$, $j = 0$: $\Box\phi = 0$ homogene Wellengleichung

$\Box A = 0$

mit $E = -\dot{A} - \nabla\phi$, $B = \nabla\times A$ gilt auch: $\Box E = 0$, $\Box B = 0$

$$\hookrightarrow \nabla\times(\underbrace{\nabla\times E}_{=-\dot{B}}) = \nabla\underbrace{(\nabla\cdot E)}_{=0} - \Delta E = -\nabla\times\dot{B} \underset{\substack{\uparrow \\ \nabla\times B = \epsilon_0\mu_0\dot{E}}}{=} -\epsilon_0\mu_0\ddot{E} \Leftrightarrow \left(\Delta - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)E = 0$$

$$\nabla\times(\nabla\times B) = \nabla\underbrace{(\nabla\cdot B)}_{=0} - \Delta B = \mu_0\epsilon_0\nabla\times\dot{E} = -\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}B \Leftrightarrow \left(\Delta - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)B = 0$$

allgemein: Skalare Wellengleichung $\Box\phi = \Delta\phi - \frac{1}{v^2}\ddot{\phi} = 0$

Bsp.: 1D : $\Delta = \frac{\partial^2}{\partial x^2}$ : Lösungsansatz: $\phi(x,t) = A\sin(kx - \omega t)$

$$\Rightarrow \phi_x = Ak\cos(kx-\omega t), \quad \phi_t = -A\omega\cos(kx-\omega t)$$

$$\phi_{xx} = -Ak^2\sin(kx-\omega t), \quad \phi_{tt} = -A\omega^2\sin(kx-\omega t)$$

$\Rightarrow$ Einsetzen: $\frac{\partial^2}{\partial x^2}\phi(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\phi(x,t) = 0$

$$\Leftrightarrow -\left(k^2 - \frac{\omega^2}{v^2}\right)A\sin(kx-\omega t) = 0 \qquad \forall x,t$$

$\Rightarrow \omega = \pm vk$ Dispersionsgleichung/-beziehung $(\lambda\nu = v)$

Wellenzahl: $k = \frac{2\pi}{\lambda}$ Kreisfrequenz: $\omega = 2\pi\nu$ Wellenlänge: $\lambda$

Phasengeschwindigkeit: $\left.\frac{dx}{dt}\right|_{\varphi=\text{const}} = \frac{\omega}{k} = v$ Periodendauer: $T = \frac{2\pi}{\omega}$ Amplitude $A$

Phase: $\varphi(x,t) := kx - \omega t$

vektorielle Wellengleichung: $\underline{A}\in\mathbb{R}^3$ : $\Box\underline{A} = \left(\Delta - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\underline{A} = 0$

spezielle Lösungen: harmonische ebene Wellen

$\underline{A}(\underline{r},t) = \underline{A}_0\sin(\underline{k}\cdot\underline{r} - \omega t)$ oder komplex: $\underline{A}(\underline{r},t) = \underline{A}_0 e^{i(\underline{k}\cdot\underline{r}-\omega t)}$

$\hookrightarrow A_{xx}^{(i)} = -k_x^2 A_0^{(i)}\sin(\underline{k}\cdot\underline{r} - \omega t)$

$A_{yy}^{(i)} = -k_y^2 A_0^{(i)}\sin(\underline{k}\cdot\underline{r} - \omega t)$

$A_{zz}^{(i)} = -k_z^2 A_0^{(i)}\sin(\underline{k}\cdot\underline{r} - \omega t)$

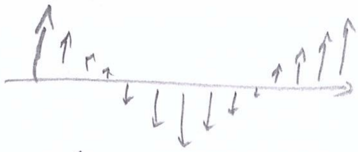
Flächen konstanter Phase:

$\varphi_0 = \underline{k}\cdot\underline{r} - \omega t$

$\nabla\varphi(\underline{r},t) = \nabla(\underline{k}\cdot\underline{r} - \omega t) = \underline{k}$ Wellenvektor

Ausbreitungsrichtung: $\frac{\underline{k}}{|\underline{k}|}$ Wellenlänge: $\lambda = \frac{2\pi}{|\underline{k}|}$

Bsp.: (i) Transversalwelle $\underline{A}_0 \perp \underline{k}$



$$\underline{\nabla}\cdot\underline{A} = \underline{A}_0\cdot\underline{\nabla}\sin(\underline{k}\cdot\underline{r}-\omega t) = \underbrace{\underline{A}_0\cdot\underline{k}}_{=0}\cos(\underline{k}\cdot\underline{r}-\omega t) = 0 \quad \text{(quellenfrei!)}$$

Lichtwelle, Seilwellen

(ii) Longitudinalwelle $\underline{A}_0 \parallel \underline{k}$


$$\underline{\nabla}\times\underline{A} = \underbrace{\underline{k}\times\underline{A}_0}_{=0}\cos(\underline{k}\cdot\underline{r}-\omega t) = 0 \quad \text{(wirbelfrei)}$$

Schallwellen

allgemein: Zerlegen in transversalen und longitudinalen Anteil

$$\underline{A}(\underline{r},t) = \underline{A}_t(\underline{r},t) + \underline{A}_l(\underline{r},t)$$

Bsp: (i) $\underline{B}$ immer transversal ($\underline{\nabla}\cdot\underline{B}=0$)

(ii) $\underline{E}$ statisch longitudinal

(iii) $\underline{E}_{\text{dynamisch}} = \underset{\substack{\uparrow \\ -\dot{\underline{A}}}}{\underline{E}_t} + \underset{\substack{\uparrow \\ -\underline{\nabla}\phi}}{\underline{E}_l}$

(Coulomb-Eichung)

allgemeine Lösung: d'Alembertsche Lösung $\underline{A}(\underline{r},t) = \sum_j \underline{F}_j(\underline{k}_j\cdot\underline{r}-\omega_j t)$

mit $\omega_j = v|\underline{k}_j|$

Anwendungen:

a) Interferenz ebener Wellen

b) Wellenpakete

c) elektromagnetische Wellen in Materie (Isolatoren)