normale Dispersion: $v_g < v_{ph} \Leftrightarrow \frac{dn}{dk}\Big|_{k_0} < 0$

anomale Dispersion: $v_g > v_{ph}$

keine Dispersion: $v_g = v_{ph}$

(c) elektromagnetische Wellen (im Isolator)

$$\Delta \underline{E} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\underline{E} = 0 \quad \text{mit } v^2 = \frac{1}{\varepsilon_0 \varepsilon \mu_0 \mu} = \frac{c^2}{n^2}$$

↳ $n = \sqrt{\varepsilon\mu}$ Brechungsindex

↳ Vakuum: $n = 1 \Rightarrow v = c = v_{ph} = v_g$

↳ in dispersiven Medien: n hängt von ω (bzw. k) ab

$$\frac{dv}{dk} = c\frac{d}{dk}\frac{1}{n} = -\frac{c}{n^2}\frac{dn}{dk} \begin{array}{l} < 0 \text{ normal} \\ > 0 \text{ anomal} \end{array}$$

$\Rightarrow v_g = v_{ph} - k_0 \frac{c}{n^2}\frac{dn}{dk}$ ⟶ $v_g > c$ möglich, aber $v_{Signal} \leq c$!

d) Fourier-Darstellung von Wellenpaketen:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk \quad , \quad \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

Fourier-Integral   Fourier-Transformierte $\mathcal{F}[f]$

$\mathcal{F}: f \to \hat{f}$

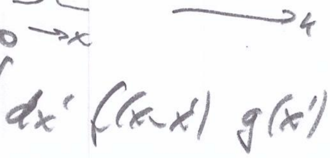
Eigenschaften:

↳ linear

↳ Multiplikationssatz $\frac{d}{dx}f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \hat{f}(k)\, \frac{d}{dx} e^{ikx}$

$$\Rightarrow \mathcal{F}\left[\frac{d}{dx}f(x)\right] = ik\hat{f}(k)$$

↳ δ-Funktion: $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}$



↳ $\hat{y}(k) = \hat{f}(k)\, \hat{g}(k) \Rightarrow y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx'\, f(x-x')\, g(x')$ Faltung

$= (g * h)(x)$

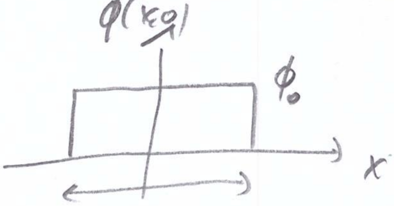
Wellenpaket:

Zunächst: $t=0$

$$\phi(x,0)=\int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \hat{\phi}(k)\, e^{ikx}$$

$$\hat{\phi}(k) = \sqrt{2\pi}\, A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, \phi(x,0)\, e^{-ikx}$$

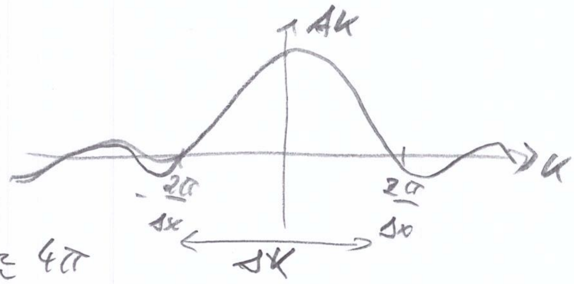
Bsp:



$$A(k) = \frac{\phi_0}{2\pi}\int_{-\Delta x/2}^{\Delta x/2} dx\, e^{-ikx} = \frac{\phi_0}{2\pi} \left.\frac{e^{-ikx}}{-ik}\right|_{-\frac{\Delta x}{2}}^{\Delta x/2}$$

$$= \frac{\phi_0}{\pi}\,\frac{e^{-ik\frac{\Delta x}{2}} - e^{ik\frac{\Delta x}{2}}}{-2ik} = \frac{\phi_0}{\pi}\,\frac{\sin\left(k\frac{\Delta x}{2}\right)}{k}$$

$$= \frac{\phi_0 \Delta x}{2\pi}\,\frac{\sin\left(k\frac{\Delta x}{2}\right)}{k\frac{\Delta x}{2}}$$



Unschärferelation $\Delta x \cdot \Delta k \approx 4\pi$

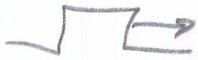
Breite im k-Raum $\approx \frac{4\pi}{\Delta x}$

$\Rightarrow$ Je schärfer im Ortsraum lokalisiert, desto breiter im k-Raum

zeitliche Entwicklung: $\phi(x,t) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{i(kx-\omega t)}$

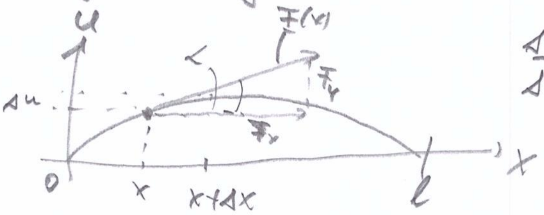
ohne Dispersion: $\omega = vk$: $\phi(x,t) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\underbrace{(x-vt)}_{\tilde{x}}} = \phi(x-vt,0)$

$\Rightarrow$ Wellenpaket behält seine Form



mit Dispersion: $\omega = v(k)\,k$: Wellenpaket fließt auseinander (Δx wächst mit t)

• Eigenschwingung einer Saite (ideal biegsam, kleine Auslenkung)



$$\frac{\Delta u}{\Delta x} = \tan\alpha \approx \sin\alpha$$

$$F_{total} = \underbrace{F_y(x+\Delta x) - F_y(x)}_{\text{rückstellende Auslenkkraft}} = F\left(\sin\alpha\big|_{x+\Delta x} - \sin\alpha\big|_x\right) \approx F\left(\frac{\Delta u}{\Delta x}\Big|_{x+\Delta x} - \frac{\Delta u}{\Delta x}\Big|_{x}\right)$$

$$\xrightarrow{\Delta x\to 0} F_{tot} = F\left(\underbrace{\frac{\partial u}{\partial x}\Big|_{x+dx}}_{\frac{\partial u}{\partial x}\big|_x + \frac{\partial u}{\partial x}\big|_x' dx} - \frac{\partial u}{\partial x}\Big|_x\right) = F\,dx\,\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2}{\partial t^2}u$$

$m = \rho q\, dx$, q: Querschnitt

$$\Rightarrow \quad u_{xx} - \frac{1}{v^2}u_{tt} = 0 \quad \text{mit} \quad v = \sqrt{\frac{F}{\rho q}} = \sqrt{\frac{\sigma}{\rho}}, \ \sigma: \text{Spannung}$$

$$\frac{\partial^2}{\partial x^2}u - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u = 0$$

Randbedingung: $u(0,t) = u(l,t) = 0$ Anfangsbed.: $u(x,0) = \varphi(x)$, $u_t(x,0) = 0$ Startbedingungen

Trennung der Variablen: $u(x,t) = f(x)\,T(t)$

aus Randbed.: $\left.\begin{array}{l} f(0)\,T(t) = 0 \\ f(l)\,T(t) = 0 \end{array}\right\} \xRightarrow{\forall t} f(0) = f(l) = 0$

→ Einsetzen: $f''\,T = \frac{1}{v^2} f\ddot{T} \xrightarrow{\forall x,t} \frac{f''}{f} = \frac{1}{v^2}\frac{\ddot{T}}{T} = -\lambda = \text{const.}$

$\Rightarrow \quad f''(x) + \lambda f(x) = 0$ Randwertproblem

$\ddot{T}(t) + \lambda v^2 T(t) = 0$ Anfangswertproblem

→ Eigenwertproblem: λ: Eigenwert

(i) $\lambda = 0$: $f'' = 0 \Rightarrow f(x) = Ax + B$
$\left.\begin{array}{l} f(0) = B \stackrel{!}{=} 0 \\ f(l) = Al = 0 \end{array}\right\} \Rightarrow f \equiv 0 \Rightarrow$ kein Eigenwert

(ii) $\lambda \neq 0$ $f'' + \lambda f = 0$

Ansatz: $f(x) = e^{\alpha x} \Rightarrow \alpha^2 e^{\alpha x} + \lambda e^{\alpha x} = 0 \Rightarrow \alpha = \pm i\sqrt{\lambda}$

$$\Rightarrow f(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x}$$
$$= a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x) \quad \text{für } \lambda > 0$$

$\hookrightarrow \lambda < 0: \quad f(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$ mit $\alpha = \sqrt{-\lambda} > 0$

$\hookrightarrow f(0) = C_1 + C_2 \overset{!}{=} 0$

$f(l) = C_1 e^{\alpha l} + C_2 e^{-\alpha l} = C_1 \underbrace{(e^{\alpha l} - e^{-\alpha l})}_{2\sinh(\alpha l) \neq 0} \overset{!}{=} 0$

$\Rightarrow C_1 = 0 \Rightarrow C_2 = 0 \Rightarrow f \equiv 0 \Rightarrow \lambda < 0$ kein Eigenwert

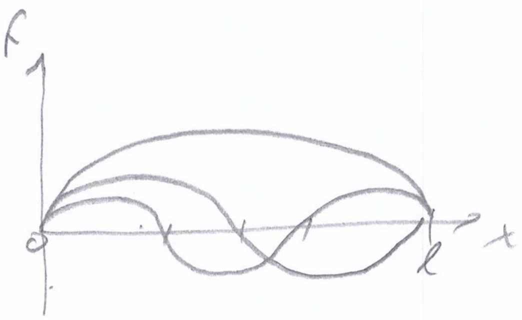
Bleibt: $\lambda > 0$:

$f(0) = a \overset{!}{=} 0$

$f(l) = b \sin(\sqrt{\lambda}\, l) \overset{!}{=} 0 \underset{b \neq 0}{\Rightarrow} \sqrt{\lambda}\, l = n\pi, \; n = 1, 2, 3 \ldots$

diskretes Spektrum: $\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad n \in \mathbb{N}$ Eigenwerte (kleinster Eigenwert $\lambda_1 = \left(\frac{\pi}{l}\right)^2$)

$\hookrightarrow f_n(x) = b_n \sin\left(\underbrace{\frac{n\pi}{l}}_{k_n} x\right)$ Eigenlösungen



Zeitabhängiger Teil der Wellengleichung: $\ddot{T} + \lambda_n v^2 T = 0$

$\Rightarrow T_n(t) = A_n \cos\left(\underbrace{\frac{n\pi}{l} v}_{\omega_n} t\right) + B_n \sin\left(\frac{n\pi}{l} v t\right)$

$\Rightarrow$ Lösg: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} f_n(x)\, T_n(t)$

$= \sum_{n=1}^{\infty} \underbrace{\left[A_n \cos\left(\frac{n\pi}{l} v t\right) + B_n \sin\left(\frac{n\pi}{l} v t\right)\right]}_{\text{stehende Welle}} \underbrace{\sin\left(\frac{n\pi}{l} x\right)}_{\text{für Randbed.}}$